2017 Kansas Mathematics Standards

Flip Book

6th Grade

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope." 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path, if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size". In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process (Click on photo to view video).

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in Grade 6

The Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students grades K-12. Below are a few examples of how these Practices may be integrated into tasks that sixth grade students complete.

Practice	Explanation and Example
1) Make Sense and Persevere in Solving Problems.	Mathematically proficient students in Grade 6 start by explaining to themselves the meaning of the problem and looking for entry points to its solution. They solve problems involving ratios and rates and discuss how they solved them. Sixth graders solve real world problems through the application of algebraic and geometric concepts. They seek the meaning of a problem and look for efficient ways to represent and solve it. They check their thinking by asking themselves, "What is the most efficient way to solve the problem?", "Does this make sense?", and "Can I solve the problem in a different way?" Example: to understand why a 20% discount followed by a 20% markup does not return an item to its original price, a 6th grader might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item prices at $1.00 or $10.00.
2) Reason abstractly and quantitatively.	Mathematically proficient students in Grade 6 represent a wide variety of real world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. Sixth graders are able to contextualize to understand the meaning of the number or variable as related to the problem. They decontextualize to manipulate symbolic representations by applying properties of operations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. Grade 6 students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students in Grade 6 construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, and graphs, tables, and other data displays. They refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. Proficient sixth graders progress from arguing exclusively through concrete referents such as physical objects and pictorial representations, to also include symbolic representations such as expressions and equations. Sixth graders can answer questions like, "How did you get that"? "Why is that true?", and "Does that always work?" Proficient 6th graders explain their thinking to others and respond to others' thinking.
4) Model with mathematics.	Mathematically proficient students in Grade 6 can apply the mathematics they know to solve problems arising in everyday life. For example, 6th graders might apply proportional reasoning to plan a school event or analyze a problem in the community. Proficient students model problem situations symbolically, graphically, tabularly, and contextually. They form expressions, equations, or inequalities from real world contexts and connect symbolic and graphical representations. Sixth graders begin to explore covariance and represent two quantities simultaneously. They use number lines to compare numbers and represent inequalities. Students in Grade 6 use measures of center and variability and data displays (i.e. box plots and histograms) to draw inferences about and make comparisons between data sets. Sixth graders connect and explain the connections between the different representations. They use all representations as appropriate to a problem context.

Practice	Explanation and Example
5) Use appropriate tools strategically.	Mathematically proficient students in Grade 6 consider the available tools (including estimation and technology) when solving a mathematical problem and decide when certain tools might be helpful. Students in 6th grade might decide to represent similar data sets using dot plots with the same scale to visually compare the center and variability of the data. They use physical objects or applets to construct nets and calculate the surface area of three dimensional figures. This practices is also related to looking for structure (SMP 7), which often results in building mathematical tools that can then be used to solve problems.
6) Attend to precision.	Mathematically proficient students in Grade 6 continue to refine their mathematical communications skills by using clear and precise language in their discussions with others and in their own reasoning. Sixth graders use appropriate terminology when referring to rates, ratios, geometric figures, data displays, and components of expressions, equations or inequalities. Students in Grade 6 are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
7) Look for and make use of structure.	Mathematically proficient students in Grade 6 routinely seek patterns or structures to model and solve problems. They recognize patterns that exist in ratio tables recognizing both the additive and multiplicative properties. Sixth graders can apply properties to generate equivalent expressions (i.e. 6 + 3x = 3(2 + x) by distributive property. They solve equations (i.e. 2c + 3 = 15, 2c = 12 by subtraction property of equality, c = 6 by division property of equality). They compose and decompose two-and three-dimensional figures to solve real world problems involving area and volume.
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students in Grade 6 use repeated reasoning to understand algorithms and make generalizations about patterns. They solve and model problems. They may notice that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ and construct other examples and models that confirm their generalizations. Students in Grade 6 connect place value and their prior work with operations to understand algorithms to fluently divide multi-digit numbers and perform all operations with multi-digit decimals. Sixth graders informally begin to make connections between covariance, rates, and representations showing the relationships between quantities.

Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others' solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information.

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include;: concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.).
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
    - a task when there is no need to have an exact answer.
    - a task when there is not enough information to get an exact answer.
    - a task to check if the answer from a calculation is reasonable.

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem~~.~~(i.e. – decomposing numbers by place value; working with properties; etc.).
- Asks students to take a complex idea and then identify and use the component parts to solve problems~~.~~ (i.e., Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm). When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size"~~.~~ (example below):

4)351 -32	3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.
31	There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.
-28 3	This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation~~.~~ i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in 6th Grade

In In Grade 6, instructional time should focus on **five** critical areas:

1. **Connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems.**
   Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.

2. **Completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers.**
   Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.

3. **Writing, interpreting, and using expressions and equations.**
   Students understand the use of variables in mathematical expressions. They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.

4. **Developing an understanding of volume and surface area of prisms.**
   Building on the Grade 5 concept of packing unit cubes to find the volume of a rectangular prism with whole number edge lengths, students develop and apply a formula to find the volume of right rectangular prisms with fractional edge lengths. Students also represent three-dimensional figures with nets and use them to find surface area of prisms.

5. **Developing understanding of statistical thinking.**
Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically. In Grade 6, two big statistical ideas are developed: measures of center and measures of variability. Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. Students recognize that a measure of variability (range and interquartile range) can also be useful for summarizing data highlighting the spread of the data rather than just the center. Two very different sets of data can have the same mean and median yet be distinguished by their variability. This leads to an informal study of the impact of outliers. Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, outliers, and symmetry, considering the context in which the data were collected.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset

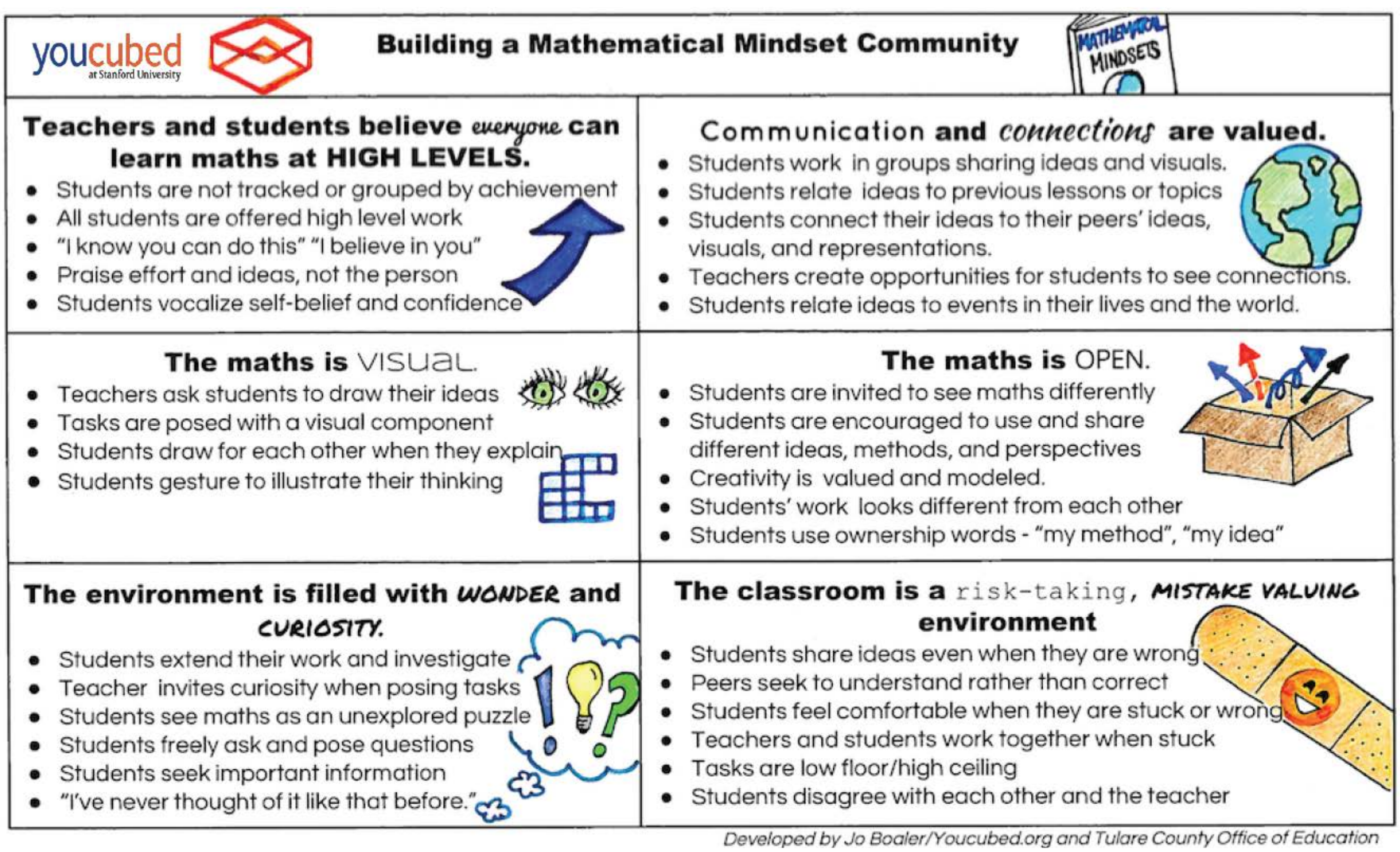
The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.



Grade 6 Content Standards Overview

Ratios and Proportional Relationships (6.RP)

A. Understand ratio concepts and use ratio reasoning to solve problems.
6.RP.1 **6.RP.2** **6.RP.3**

The Number System (6.NS)

A. Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
6.NS.1

B. Compute fluently (efficiently, accurately, and flexibly) with multi-digit numbers and find common factors and multiples.
6.NS.2 **6.NS.3** **6.NS.4**

C. Apply and extend previous understandings of numbers to the system of **rational numbers**
6.NS.5 **6.NS.6** **6.NS.7** **6.NS.8**

Expressions and Equations (6.EE)

A. Apply and extend previous understandings of arithmetic to algebraic expressions.
6.EE.1 **6.EE.2** **6.EE.3**

B. Reason about and solve one-variable equations and inequalities.
6.EE.4 **6.EE.5** **6.EE.6** **6.EE.7**

C. Represent and analyze quantitative relationships between dependent and independent variables.
6.EE.8

Geometry (6.G)

A. Solve real-world and mathematical problems involving area, surface area, and volume.
6.G.1 **6.G.2** **6.G.3** **6.G.4**

Statistics and Probability (6.SP)

A. Develop concepts of statistical measures of center and variability and an informal understanding of **outlier**.
6.SP.1 **6.SP.2** **6.SP.3**

B. Summarize and describe distributions.
6.SP.4 **6.SP.5**

Standards for Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Domain: Ratios and Proportional Relationships (RP)

▶ ***Cluster A:*** *Understand ratio concepts and use ratio reasoning to solve problems.*

Standard: 6.RP.1

Use ratio language to describe a relationship between two quantities. Distinguish between part-to-part and part-to-whole relationships. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."* **(6.RP.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: **(6.RP.1-3)**

This cluster is connected to:

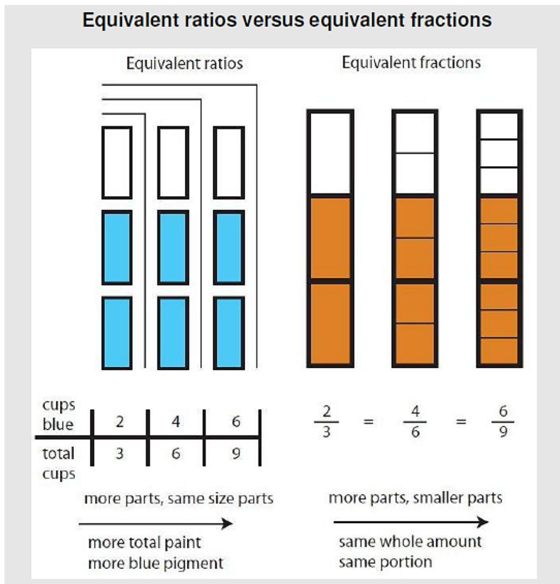
- Grade 6 ***connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems.***
- In Grade 6, students develop the foundational understanding of ratio and proportion that will be extended in Grade 7 to include scale drawings, slope and real-world percent problems.

Explanations and Examples:

A ratio is a comparison of two quantities, or measures, which can be written as ***a* to *b***, $\frac{a}{b}$, or ***a*:*b***. The comparison can be **part-to-whole** (ratio of guppies to all fish in an aquarium) or **part-to-part** (ratio of guppies to goldfish). This is very different from the work students just completed with fractions in third through fifth grades. Students have been told that the denominator represents the total number of parts in a whole. Since the same representation of fractions ($\frac{a}{b}$) can be used for ratios, this is potentially very confusing. It must be clear when the ratio is comparing part-to-whole and when the ratio is comparing part-to-part. **Ratios are relations that involve multiplicative reasoning** – this is a critical concept of ratio reasoning that cannot be over emphasized.

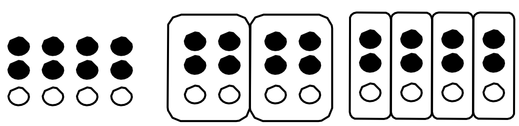
Students need to understand each of these ratios when expressed in the following forms: $\frac{6}{15}$, 6 to 15, or 6:15. Students will need to understand that these values can be simplified to, $\frac{2}{5}$, 2 to 5, or 2:5; *however, students will need to understand how the simplified values relate to the original numbers*. **Students have been working with equivalent fractions since third grade but they have not been expected to "reduce" fractions or to put them into "simplest form."** Since third through fifth grade students are working on understanding the concept of fractions and how to compute them, it was not necessary to increase the cognitive load to make students reduce or simplify fractions. This now becomes part of the expectations for students in middle school. These students have been expected to know and understand equivalent forms so build on this understanding to simplify fractions.

Look at this graphic from the Ratios and Proportional Relationships Learning Progression to see an example distinguishing between equivalent fractions and equivalent ratios:



Ratios

A comparison of 8 black circles to 4 white circles can be written as the ratio of 8:4 and can be regrouped into 4 black circles to 2 white circles (4:2) and 2 black circles to 1 white circle (2:1).



Students should be able to identify all these ratios and describe them using "For every…, there are…"

This standard specifically addresses **part-to-whole** and **part-to-part ratios**, but there are two other types of ratios (Van de Walle, Bay-Williams, Lovin, & Karp, 2014).

- **Ratios as Quotients** – "For example, if you can buy 4 kiwis for $1.00, then ratio of money for kiwis is $1.00 to 4 kiwis. The cost per kiwi ($0.25) is the unit rate" (Van de Walle et al, 2014, p. 199).
- **Ratios as Rates** – "Rates involve two different units and how they relate to each other (miles per gallon, passengers per busload, and roses per bouquet). Relationships between two units of measure are also rates – for example, inches per foot, milliliters per liter, and centimeters per inch. A rate represents an infinite set of equivalent ratios" (Van de Walle et al, 2014, p. 199).

Example Task:
A restaurant worker used 5 loaves of wheat bread and 2 loaves of rye bread to make sandwiches for an event.

- Write a ratio that compares the number of loaves of rye bread to the number of loaves of wheat bread. What does that ratio mean?
- Describe what the ratio 7:2 means in terms of the loaves of bread used for the event.

Sample Response:

- 2:5 – this ratio means "for every 2 loaves of rye bread, there are 5 loaves of wheat bread" which is a part-to-part ratio.
- 7:2 is the ratio of the total number of loaves of bread to the number of loaves of rye bread which is a whole-to-part ratio.

Instructional Strategies: 6.RP.1

Students develop the understanding that ratio is a comparison of two numbers or quantities. Ratios can be written as a part-to-whole or a part-to-part relationship. Because of their work with fractions in earlier grades students are often comfortable with part-to-whole relationships. However, 6th grade students will likely need help in differentiating between the two types of ratios.

Part-to-part ratios are used to compare two parts. For example, the number of girls in the class (12) compared to the number of boys in the class (16) is the ratio the ratio 12 to 16. This form of ratios is often used to compare the event that can happen to the event that cannot happen. Rates, a relationship between two units of measure, can be written as ratios, such as miles per hour, ounces per gallon and students per bus.
For example, 3 cans of pudding cost $2.48 at Store A and 6 cans of the same pudding costs $4.50 at Store B. Which store has the better buy on these cans of pudding? Various strategies could be used to solve this problem:

- A student can determine the unit cost of 1 can of pudding at each store and compare.
- A student can determine the cost of 6 cans of pudding at Store A by doubling $2.48.
- A student can determine the cost of 3 cans of pudding at Store B by taking $\frac{1}{2}$ of $4.50.

Proportional reasoning is a process that requires intensive instruction and relevant and authentic practice. It does not develop over time on its own. Therefore, the standards are structured in such a way as to develop a student's proportional reasoning over time. In 6th grade the focus is on ratios where they develop an understanding of the relationship between two quantities. The gaining of this understanding will begin to cultivate their multiplicative thinking which will culminate into proportional reasoning in 7th grade. Examples with ratio and proportion must involve measurements, prices and geometric contexts, as well as rates, such as miles per hour or portions per person, within contexts that are relevant to sixth graders. Experience with proportional and non-proportional relationships, comparing and predicting ratios, and **relating unit rates to previously learned unit fractions** will facilitate the development of proportional reasoning. Although algorithms provide efficient means for finding solutions to proportions (cross product algorithm), *often they* ***will not*** aid in the development of proportional reasoning. ***Delaying the introduction of rules and algorithms will encourage thinking about multiplicative situations instead of indiscriminately applying rules.***

Multiplicative reasoning is used when finding the missing element in a proportion. For example, use 2 cups of syrup to 5 cups of water to make fruit punch. If 6 cups of syrup are used to make punch, how many cups of water are needed?

$$\frac{2}{5} = \frac{6}{x}$$

- Recognize that the relationship between 2 and 6 is 3 times; $2 \cdot 3 = 6$
- To find *x*, the relationship between 5 and *x* must also be 3 times. $3 \cdot 5 = x$, therefore, $x = 15$.

$$\frac{2}{5} = \frac{6}{15}$$

The final proportion.

Other ways to illustrate ratios that will help students see the relationships follow. Begin written representation of ratios with the words "out of" or "to" before using the symbolic notation of the colon and then the fraction bar; for example, 3 out of 7, 3 to 7, 3:7 and $\frac{3}{7}$.

Labeling units helps students organize the quantities when writing proportions.

$$\frac{3\ eggs}{2\ cups\ of\ flour} = \frac{z\ eggs}{8\ cups\ of\ flour}$$

Using hue/color intensity is a visual way to examine ratios of part-to-part. Students can compare the intensity of the color green and relate that to the ratio of colors used. For example, have students mix green paint into white paint in the following ratios: 1 part green to 5 parts white, 2 parts green to 3 parts white, and 3 parts green to 7 parts white. They can then compare the green color intensity with their ratios.

Common Misconceptions:

Fractions and ratios may represent different comparisons. Fractions can express a part-to- whole comparison, but ratios can express a part-to-whole comparison OR a part-to-part comparison which can be written as: *a* to *b*, $\frac{a}{b}$, or *a*:*b*.

Even though ratios and fractions express a part-to-whole comparison, the addition of ratios and the addition of fractions are *distinctly different procedures*. When adding ratios, the parts are added, the wholes are added and then the total part is compared to the total whole. For example, (2 out of 3 parts) + (4 out of 5 parts) is equal to 6 parts out of 8 total parts (6 out of 8) if the parts are equal. When dealing with fractions, the procedure for addition is based on a common denominator: $\left(\frac{2}{3}\right) + \left(\frac{4}{5}\right) = \left(\frac{10}{15}\right) + \left(\frac{12}{15}\right)$ which is equal to $\left(\frac{22}{15}\right)$. Therefore, the addition process for ratios and for fractions is distinctly different.

Students may confuse mathematical terms such as ratio, rate, unit rate and percent.

Students may not understand the difference between an additive relationship and a multiplicative relationship.

Resources/Tools:

For detailed information, see Learning Progressions Ratio and Proportional Relationships Gr-6-7

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource requires membership access check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership this would be a valuable resource to request.

- "Constant Dimensions" NCTM Illuminations. Students measure length and width of rectangle and work to discover the ratio of length to width regardless of the use of non-standard or standard units.

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

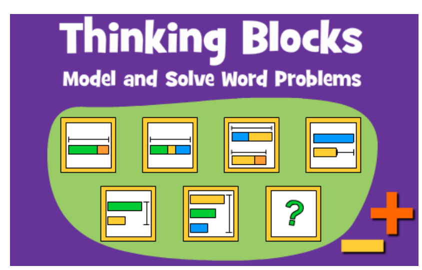
- 6.RP.A
  - Ratio of boys to girls
  - Climbing the steps of El Castillo
- 6.RP.A.1
  - Games at Recess
  - The Escalator, Assessment Variation
  - Bag of Marbles
  - Evaluating Ratio Statements
  - Apples to Apples
  - Many Ways to Say It
  - Representing a Context with a Ratio

Nrich Mathematics:

- Mixing Paints
- Mixing More Paints
- Ratios and Dilutions
- Ratio Pairs 2
- Ratio Pairs 3
- The Golden Ratio
- Rod Ratios
- Ratio or Proportion?
- Ratios, Proportions, and Rates of Change problems – Short Problems

Thinking Blocks:

This website houses several online bar method tools that can be used to solve ratio problems. Click on the hyperlinked word "Ratios" to access this website, then scroll down to Ratios.



YouTube Video clips:

- What is a ratio?
- What is a ratio? 2

Open Middle:

- Finding Equivalent Ratios

Domain: Ratios and Proportional Reasoning (RP)

▶ **Cluster A:** *Understand ratio concepts and use ratio reasoning to solve problems*.

Standard: 6.RP.2

Use unit rate language ("for each one", "for every one" and "per") and unit rate notation to demonstrate understanding the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a: b$ $with$ $b \neq 0$, *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is* $\frac{3}{4}$ *cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."* (Expectations for unit rates in this grade are limited to non-complex fractions.) **(6.RP.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections: See 6.RP.1

Explanations and Examples: 6.RP.2

A rate is a ratio where two measurements are related to each other. When discussing measurement of different units, the word rate is used rather than ratio. Understanding rate, however, is complicated and there is no universally accepted definition. When using the term rate, contextual understanding is critical. Students need many opportunities to use models to demonstrate the relationships between quantities before they are expected to work with rates numerically.

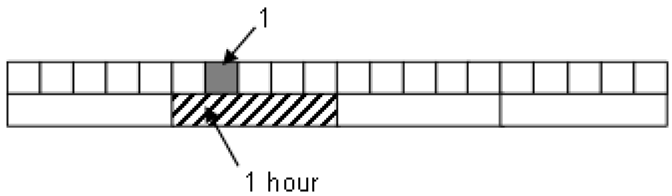
A unit rate expresses a ratio as part-to-one or one unit of another quantity. Students understand the unit rate from various contextual situations. For example, if there are 2 cookies for 3 students, each student receives $\frac{2}{3}$ of a cookie, so the unit rate is 2:1. If a car travels 240 miles in 4 hours, the car travels 60 miles per hour (60:1).

Students will often use unit rates to solve missing value problems. Cost per item or distance per time unit are common unit rates, however, students should be able to flexibly use unit rates to name the amount of either quantity in terms of the other quantity. Students will begin to notice that related unit rates are reciprocals as in the first example. It is not intended that this be taught as an algorithm or rule because at this level, students should primarily use reasoning to find these unit rates.

In Grade 6, students are not expected to work with unit rates expressed as complex fractions. The numerator and/or the denominator of the original ratio will not be fractional parts.

Examples:
On a bicycle you can travel 20 miles in 4 hours. What are the unit rates in this situation? (The distance you can travel in 1 hour and the amount of time required to travel 1 mile).

Sample Solution: You can travel 5 miles in 1 hour written as $\frac{5\,mi}{1\,hr}$ and it takes $\frac{1}{5}$ of an hour to travel each mile. Students can represent the relationship between 20 miles and 4 hours.



Resources/Tools:
Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.RP.A.2
  - Mangos for Sale
  - Price per pound and pounds per dollar
  - Riding at a Constant Speed, Assessment Variation
  - The Escalator, Assessment Variation
  - Hippos Love Pumpkins
  - Ticket Booth
  - Equivalent Ratios and Unit Rates

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- "Finding Our Top Speed", NCTM Illuminations. The discussions sets the stage for travel in the solar system. Students work with time and distance and plot data.
- What's Your Rate

Utah Education Network Lesson: Trundle Wheel

Dan Meyer's 3 Act Tasks

- Partial Product

Common Misconceptions: See 6.RP.1

Domain: Ratios and Proportional Reasoning (RP)

▶ **Cluster A:** *Understand ratio concepts and use ratio reasoning to solve problems.*

Standard: 6.RP.3

Use ratio and rate reasoning to solve real-world and mathematical problems, *(e.g. by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or using calculations.)*

6.RP.3a. Make tables of equivalent ratios relating quantities with whole-number measurements, find the missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?* **(6.RP.3a) (6.RP.3b)**

6.RP.3b. Find a percent of a quantity as a rate per 100 *(e.g. 30% of a quantity means* $\frac{30}{100}$ *times the quantity*); solve problems involving finding the whole, given a part and the percent. **(6.RP.3c)**

6.RP.3c. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. **(6.RP.3d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 6.RP.1

Explanations and Examples: Grade 6.RP.3

Fractions and percent are examples of part-to-whole ratios. Fractions are written as the part being identified compared to the whole amount. A percent is the part identified compared to the whole (100). Provide students with multiple examples of ratios, fractions and percent of this type. For example, the number of girls in the class (12) to the number of students in the class (28) is the ratio 12 to 28.

Percent is often taught in relationship to learning fractions and decimals. This cluster indicates that percent is to be taught as a special type of rate. Provide students with opportunities to find percent in the same ways they would solve rates and proportions.

The **6.RP.3a** specifically refers to using tables. Using the information in the table, find the number of yards in 24 feet.

Feet	3	6		9	15	24
Yards	1	2		3	5	?

There are several strategies that students could use to determine the solution to this problem.

- Add quantities from the table to total 24 feet (9 feet and 15 feet); therefore the number of yards must be 8 yards (3 yards and 5 yards).
- Use multiplication to find 24 feet:
    1) $3\,ft \cdot 8 = 24\,ft$; therefore $1\,yd \cdot 8 = 8\,yds$
    2) $6\,ft \cdot 4 = 24\,ft$; therefore $2\,yds \cdot 4 = 8\,yds$

Example

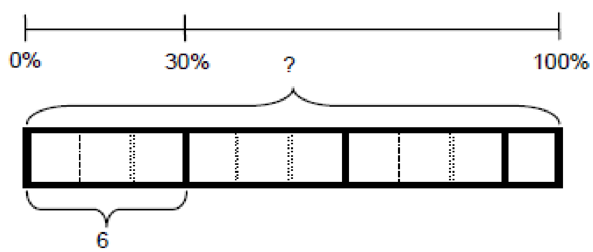
Compare the number of black to white circles. If the ratio remains the same, how many black circles will you have if you have 60 white circles?

● ● ● ● ○ ○ ○

Black	4	40	20	60	?
White	3	30	15	45	60

Example

If 6 is 30% of a value, what is that value?



Example

A credit card company charges 17% interest on any charges not paid at the end of the month.

Make a ratio table to show how much the interest would be for several amounts. If your bill totals $450 for this month, how much interest would you have to pay if you let the balance carry to the next month? Show the relationship on a graph and use the graph to predict the interest charges for a $300 balance.

Charges	$1	$50	$100	$200	$450
Interest	$0.17	$8.50	$17	$34	?

The 5th grade 5.OA.3 standard from the 2010 standards was moved here as an instructional strategy for 6.RP.3a. Students are to plot pairs of values on the coordinate plane in standard 6.RP.3a and make comparisons between the ratios.

Examples for Grade 6.RP.3a:
Ratios and rates can be used in ratio tables and graphs to solve problems. Previously, students have used additive reasoning in tables to solve problems. To begin the shift to proportional reasoning, students need to begin using multiplicative reasoning. To aid in the development of proportional reasoning the cross-product algorithm is ***not*** expected at this level. When working with ratio tables and graphs, whole number measurements are the expectation for this standard.

For example, At Books Unlimited, 3 paperback books cost $18. What would 7 books cost? How many books could be purchased with $54. To find the price of 1 book, divide $18 by 3. One book is $6. To find the price of 7 books, multiply $6 (the cost of one book times 7 to get $42. To find the number of books that can be purchased with $54, multiply $6 times 9 to get $54 and then multiply 1 book times 9 to get 9 books.

Students use ratios, unit rates and multiplicative reasoning to solve problems in various contexts, including measurement, prices, and geometry. Notice in the table below, a multiplicative relationship exists between the numbers both horizontally and vertically.

Number of Books	Cost
1	6
3	18
7	42
9	54

Students use tables to compare ratios. Another bookstore offers paperback books at the prices below.
Which bookstore has the best buy? Explain how you determined your answer.

Number of Books	Cost
4	20
8	40

To help understand the multiplicative relationship between the number of books and cost, students write equations to express the cost of any number of books. Writing equations is foundational for work in 7th grade. For example, the equation for the first table would be $c = 6n$.

The numbers in the table can be expressed as ordered pairs (number of books, cost) and plotted on a coordinate plane. Students should be able to plot ratios as ordered pairs.

Using ratio tables develops the concept of proportion. By comparing equivalent ratios, the concept of proportional thinking is developed and many problems can be solved with relative ease.

Store A		Store B	
3 cans	6 cans	6 cans	3 cans
$2.48	$4.96	$4.50	$2.50

Students should also solve real-life problems involving measurement units that need to be converted. Representing these measurement conversions with models such as ratio tables, t-charts or double number line diagrams will help students internalize the size relationships between same system measurements and relate the process of converting to the solution of a ratio.

Students recognize the use of ratios, unit rate and multiplication in solving problems, which could allow for the use of fractions and decimals. The ratio tables above use unit rate by determining the cost of one book. However, ratio tables can be used to solve problems without the use of a unit rate.

For example, in trail mix, the ratio of cups of peanuts to cups of chocolate candies is 3 to 2.
How many cups of chocolate candies would be needed for 9 cups of peanuts?

Peanuts	Chocolate
3	2

One possible way to solve this problem is to recognize that 3 cups of peanuts times 3 will give 9 cups. The amount of chocolate will also increase at the same rate (3 times) to give 6 cups of chocolate. Students could also find the number of cups of chocolate candies for 1 cup of peanuts by dividing both sides of the table by 3, giving $\frac{2}{3}$ cup of chocolate for each cup of peanuts. To find the amount of chocolate needed for 9 cups of peanuts, students multiply the unit rate by nine $\left(9 \times \frac{2}{3}\right)$giving 6 cups of chocolate.

<u>Examples for Grade 6.RP.3 b & c</u>:
This is the students' first introduction to percent. Percent is a rate *per 100*. Models, such as percent bars or $10 \text{ x } 10$ grids should be used to model percents. Students use percentages to find the part when given the percent, by recognizing that the whole is being divided into 100 parts and then taking a part of them (the percent). For example, to find 40% of 30, students could use a $10 \text{ x } 10$ grid to represent the whole (or 30). If the 30 is divided into 100 parts, the rate for one block is 0.3. Forty percent would be 40 of the blocks, or 40×0.3, which equals 12.Students also find the whole, given a part and the percent. For example, if 25% of the students in Mrs. Rumford's class like chocolate ice cream, then how many students are in Mrs. Rumford's class if 6 like chocolate ice cream? Students can reason that if 25% is 6 and 100% is 4 times the 25%, then 6 times 4 would give 24 students in Mrs. Rumford's class.

A ratio can be used to compare measures of two different types, such as inches per foot, milliliters per liter and centimeters per inch. Students recognize that a conversion factor is a fraction equal to 1 since the quantity described in the numerator and denominator is the same. For example 12 inches is a conversion 1 foot factor since the numerator and denominator name the same amount. Since the ratio is equivalent to 1, the identity property of multiplication allows an amount to be multiplied by the ratio. Also, the value of the ratio can also be expressed as $\frac{1\ foot}{12\ inches}$.

Students use ratios as conversion factors and the identity property for multiplication to convert ratio units. For example, how many centimeters are in 7 feet, given that $1\ inch = 2.54\ cm$?

$$7\ feet \times \frac{12\ inches}{1\ foot} \times \frac{2.54\ cm}{1\ inch} = 7\ feet \times \frac{12\ inches}{1\ \cancel{foot}} \times \frac{2.54\ cm}{1\ \cancel{inch}} = 7 \times 12 \times 2.54\ cm = 213.36\ cm$$

Note: Conversion factors will be given. Conversions can occur both between and across the metric and English system. Estimates are not expected.

Resources/Tools:

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- "Bagel Algebra", NCTM Illuminations. Students get to think about solving real-world problems symbolically as they interpret results to understand the bagel shop owner's point.
- Shopping Mall Math
- Big Math and Fries

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.RP.A.3
    - Voting for Two, Variation 4
    - Mixing Concrete
    - Voting for Three, Variation 1
    - Voting for Three, Variation 2
    - Voting for Three, Variation 3
    - Converting Square Units
    - Security Camera
    - Dana's House
    - Kendall's Vase - Tax
    - Currency Exchange
    - Friends Meeting on Bicycles
    - Running at a Constant Speed
    - Jim and Jesse's Money
    - Fruit Salad
    - Riding at a Constant Speed, Assessment Variation
    - Pennies to heaven
    - Gianna's Job
    - Running at a Constant Speed
    - Overlapping Squares
    - Which Detergent is a Better Buy?
    - Hunger Games vs Divergent
    - Fizzy Juice
    - Party Planning
    - Same and Different
    - Baking Bread 2
    - Constant Speed
    - Perfect Purple Paint 1
- 6.RP.A.3.a
    - Walk-a-thon 1
- 6.RP.A.3.b
    - Data Transfer
- 6.RP.A.3.c
    - Shirt Sale
    - Anna in D.C.
    - Exam Scores
    -
- 6.RP.A.3.d
    - Unit Conversions
    - Speed Conversions
    - Simple Unit Conversion Using Ratio Reasoning

Nrich Mathematics:

- Mixing Lemonade

Estimation 180:

- Yogurt Proportions
- Candle Eyes

Open Middle:

- Percentages
- Interpreting Percentages
- Related Percentages
- Percent on a Linear Model 3
- Percent on a Linear Model 4
- Percent on a Linear Model 5

Mathematics Assessment Resource Service:

- Security Camera
- Maximizing Profit: Selling Soup
- Sharing Costs Equitably: Traveling to School

Mathalicious:

- New-Tritional Info

Dan Meyer's 3 Act Tasks:

- Nana's Paint Mix-Up
- Neptune
- Split Time
- Leaky Faucet
- Coke vs Sprite
- Sugar Packets

Common Misconceptions: See 6.RP.1

Often there is a misunderstanding that a percent is always a natural number less than or equal to 100. Provide examples of percent amounts that are greater than 100%, and percent amounts that are less 1%.

Domain: The Number System (NS)

▶ **Cluster A:** *Apply and extend previous understands of multiplication and division to divide fractions by fractions.*

Standard: **6.NS.1**

6.NS.1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, requiring multiple exposures connecting various concrete and abstract models. **(6.NS.1)** (Number System 6–8 and High School Number Progression Pg. 5 - 6.)

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Grade 6 - Complete understanding of division of fractions and extend the notion of the number to the system of rational numbers, which includes negative numbers.
- This cluster continues the work from Grade 5 in the domains Number and Operations in Base Ten and Number and Operations – Fractions.
- In Grade 7, this cluster will be extended in The Number System to rational numbers and in Ratios and Proportional Reasoning.

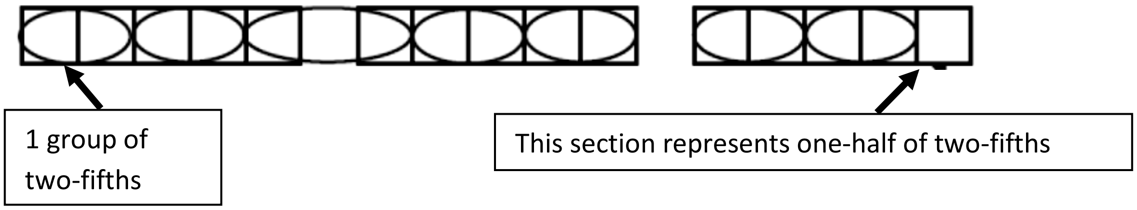
Explanations and Examples: 6.NS.1

In 5th grade students divided whole numbers by unit fractions. Students continue this understanding by using visual models and equations to divide whole numbers by fractions and fractions by fractions to solve word problems.

Students understand that a division problem such as $3 \div \frac{2}{5}$ is asking, "How many $\frac{2}{5}$s are in 3?" They should connect this to division with whole numbers. $10 \div 2$ is asking, "How many 2s are in 10?"

One possible visual model would begin with three wholes and divide each into fifths. There are 7 groups of two-fifths in the three wholes. However, one-fifth remains. Since one-fifth is half of a two-fifths group, there is a remainder of $\frac{1}{2}$. Therefore, $3 \div \frac{2}{5} = 7\frac{1}{2}$, meaning there are $7\frac{1}{2}$ groups of two-fifths.

Students interpret the solution, explaining how division by fifths can result in an answer with halves. They need to understand that the half is not half of a whole but half of two-fifths.



Explanations and Examples: 6.NS.1

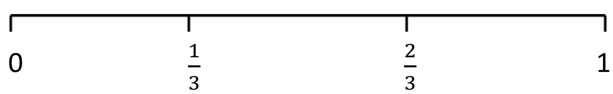
Students should write contextual problems for fraction division problems. Frequently students can solve "naked" computation problems but struggle to understand what operation(s) are to be used when confronted with a word problem (a problem in context). Allowing multiple opportunities for these types of problems and then having students write their own contextual problems is essential.

For example, the problem, $\frac{2}{3} \div \frac{1}{6}$ can be illustrated with the following word problem:

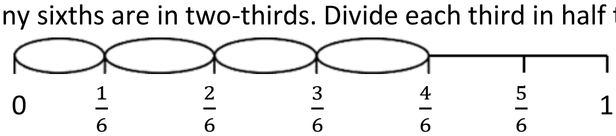
> Susan has $\frac{2}{3}$ of an hour left to make cards. It takes her about $\frac{1}{6}$ of an hour to make each card.
> About how many can she make in the time she has?

This problem can be modeled using a number line.

1. Start with a number line divided into thirds.



2. We want to know how many sixths are in two-thirds. Divide each third in half to create sixths.



3. Each circled part represents $\frac{1}{6}$. There are four of those sixths in two-thirds; therefore, Susan can make 4 cards.

Contexts and visual models can help students to understand quotients of fractions and begin to develop the relationship between multiplication and division. Model development can be facilitated by building from familiar scenarios with whole or friendly number dividends or divisors. Computing quotients of fractions build upon and extends student understandings developed in Grade 5. Students make drawings, model situations with manipulatives, or manipulate computer generated models.

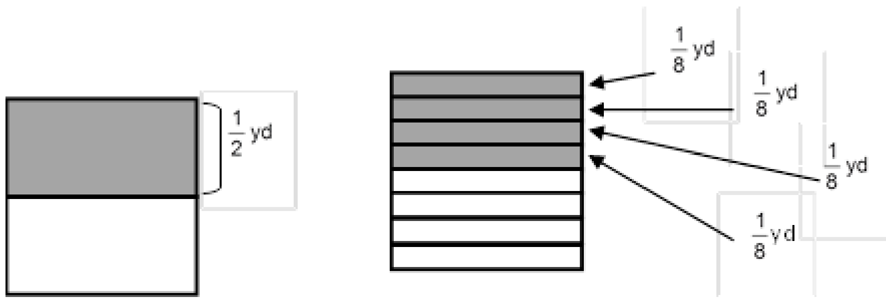
Examples:

3 people share $\frac{1}{2}$ pound of chocolate. How much of a pound of chocolate does each person get?

Solution: Each person gets $\frac{1}{6}$ lb. of chocolate.

Manny has $\frac{1}{2}$ yard of fabric to make book covers. Each book is made from $\frac{1}{8}$ yard of fabric. How many book covers can Manny make?

Solution: Manny can make 4 book covers.



Represent $\frac{1}{2} \div \frac{2}{3}$ in a problem context and draw a model to show your solution.

Context: You are making a recipe that calls for $\frac{2}{3}$ cup of yogurt. You have $\frac{1}{2}$ cup of yogurt from a snack pack. How much of the recipe can you make?

Explanation of Model below:

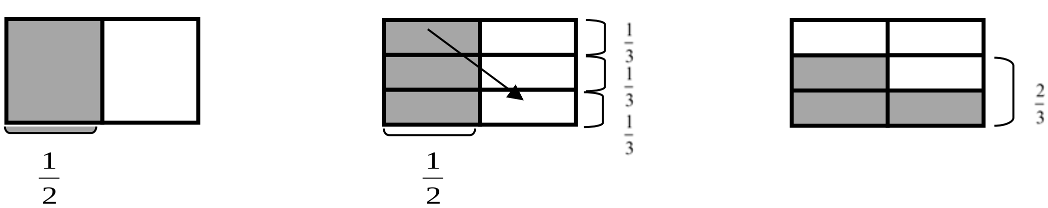
The first model shows $\frac{1}{2}$ cup. The shaded squares in all three models show $\frac{1}{2}$ cup.

The second model shows $\frac{1}{2}$ cup and also shows $\frac{1}{3}$ cups horizontally.

The third model shows $\frac{1}{2}$ cup moved to fit in only the area shown by $\frac{2}{3}$ of the model.

$\frac{2}{3}$ is the new referent unit (whole) .

3 out of the 4 squares in the $\frac{2}{3}$ portion are shaded. A $\frac{1}{2}$ cup is only $\frac{3}{4}$ of a $\frac{2}{3}$ cup portion, so you can only make $\frac{3}{4}$ of the recipe.

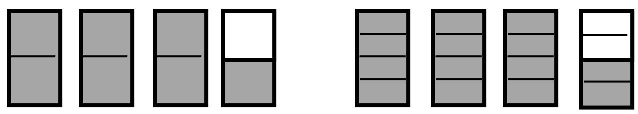


Instructional Strategies:

Computation with fractions is best understood when it builds upon the familiar understandings of whole numbers and is paired with visual representations. Solve a simpler problem with whole numbers, and then use the same steps to solve a fraction divided by a fraction. Look at the problem through the lens of "How many groups?" or "How many in each group?" helps visualize what is being solved.

For example: $12 \div 3$ means, "How many groups of three would make 12?" Or, "How many in each of 3 groups would make 12?" So, $\frac{7}{2} \div \frac{1}{4}$ can be solved the same way. "How many groups of $\frac{1}{4}$ make $\frac{7}{2}$?"
Or, "How many objects in a group when $\frac{7}{2}$ fills one fourth?"

Creating the picture that represents this problem assists in seeing the problem and making it easier to prove the solution.



<u>Teaching "invert and multiply" without developing an understanding of why it works first, leads to confusion as to when to apply the shortcut.</u>

Learning how to compute fraction division problems is one part, being able to relate the problems to real-world situations is important. Providing opportunities to create stories for fraction problems or writing equations for situations is <u>essential</u>.

Resources/Tools:

For detailed information, see Learning Progressions for NS

Georgia Department of Education:

- **"Cupid Targets Fractions and Recipes "**: Students work with fractions in a real world setting to calculate proportions of recipe and justify their answers

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.A.1
  - Baking Cookies
  - Video Game Credits
  - Making Hot Cocoa, Variation 1
  - How Many Containers in One Cup / Cups in One Container?
  - Running to School, Variation 2
  - Making Hot Cocoa, Variation 2
  - Drinking Juice, Variation 2
  - Drinking Juice, Variation 3
  - Traffic Jam
  - How many _______ are in. . . ?
  - Standing in Line
  - Running to School, Variation 3
  - Dan's Division Strategy
  - Cup of Rice

Open Middle:

- Fraction Division
- Fraction Quotient Closest to $\frac{4}{11}$
- Dividing Fractions
- Dividing Fractions 2
- Dividing Mixed Numbers

Mathematics Assessment Resource Service:

- Interpreting Multiplication and Division
- Translating between Fractions, Decimals, and Percent

Thinking Blocks:

- Fractions – Multiplication and Division

Dan Meyer's 3 Act Tasks:

- Nana's Lemonade

Common Misconceptions: 6.NS.1

Students may believe that dividing by $\frac{1}{2}$ is the same as dividing in half. Dividing by half means to find how many $\frac{1}{2}$ s there are in a quantity, whereas, dividing in half means to take a quantity and split it into two equal parts.

So 7 divided by $\frac{1}{2} = 14$ and 7 divided in half equals $3\frac{1}{2}$.

Students may incorrectly model division of fractions.
Students may not understand that larger negative numbers are smaller in value.
Students may confuse the absolute value symbol with the number one.

Domain: The Number System (NS)

● **Cluster B:** *Compute fluently with multi-digit numbers and find common factors and multiples.*

Standard: **6.NS.2**

Fluently (efficiently, accurately, and flexibly) divide multi-digit numbers using an efficient algorithm. **(6.NS.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Grade 6 - **Complete understanding of division of fractions and extend the notion of number to the system of rational numbers.**
- Other Grade 6 clusters within The Number System domain. This marks the final opportunity for students to demonstrate fluency with the four operations with whole numbers and decimals.
- Grade 7 will extend these learnings in The Number System and in Expressions and Equations.

Explanations and Examples:

Procedural fluency is defined as "skill in carrying out procedures *flexibly, accurately, efficiently and appropriately*". In the elementary grades, students were introduced to division through concrete models and various strategies to develop an **understanding** of this mathematical operation (limited to 4-digit numbers divided by 2-digit numbers). In 6th grade, students become fluent in long division using an efficient algorithm. This understanding is foundational for work with fractions and decimals in 7th grade.

Students are expected to fluently and accurately divide multi-digit whole numbers. Divisors can be any number of digits at this grade level.

As students divide they should continue to use their **understanding of place value** to describe what they are doing. When using the standard algorithm, students' language should reference place value.

For example, when dividing 32 into 8456, they should think, "There are 200 thirty-twos in 8456 so the digit 2 will be put in the hundreds place in the quotient" and then they should write 6400 beneath the 8456 rather than only writing 64. The rest of the thinking for dividing this follows in the chart. This illustrates how students need to be using place value understanding when dividing. This understanding is important in developing true number sense.

2 32)8456	There are 200 thirty-twos in 8456.
2 32)8456 −6400 2056	200 times 32 is 6400. 8456 minus 6400 is 2056.
26 32)8456 −6400 2056	There are 60 thirty-twos in 2056.
26 32)8456 −6400 2056 −1920 136	60 times 32 is 1920. 2056 minus 1920 is 136.
264 32)8456 −6400 2056 −1920 136 −128	There are 4 thirty-twos in 136. 4 times 32 is 128.
264 32)8456 −6400 2056 −1920 136 −128 8	The remainder is 8. There is not a full thirty-two in 8; there is only part of a thirty-two in 8. This can be written as $\frac{8}{32}$ or $\frac{1}{4}$. There is $\frac{1}{4}$ of a thirty-two in 8. 8456 = 264 * 32 + 8

Here's how a student could reason through another division problem using number sense:

$$1{,}716 \div 161$$

There are one hundred 16's in 1,716.
Ten groups of 16 is 160.
That's too big.
Half of that is 80, which is five groups.
I know that two groups of 16's is 32.
I have 4 left over.

$1{,}716 - 1{,}600$	100
$116 - 80$	5
36-32	2
4	

So the answer is 107 with a remainder of 4.

Instructional Strategies: 6.NS.2

As students studied whole numbers in the elementary grades, a foundation was laid in the conceptual understanding of each operation. Discovering and applying multiple strategies for computing creates connections which evolve into the proficient use of standard algorithms. Fluency with an algorithm denotes an ability that is efficient, accurate, appropriate and flexible. Division was introduced conceptually in Grade 3, as the inverse of multiplication. In Grade 4, division continued using place-value strategies, properties of operations, and the relationship with multiplication, area models, and rectangular arrays to solve problems with one digit divisors. In Grade 6, fluency with the algorithms for division and all operations with decimals was developed.

Fluency is something that develops over time; practice should be given over the course of the year as students solve problems related to other mathematical studies. Opportunities to determine when to use paper-pencil algorithms, mental math, or a computing tool is a necessary skill and should be provided in problem solving situations.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.B.2
  - Interpreting a Division Computation
  - How Many Staples?
  - Batting Average

Nrich Mathematics:

- The Remainders Game

Mathematics Assessment Resource Service:

- Using Standard Algorithms for Number Operations

Domain: The Number System (NS)

● **Cluster B:** *Compute fluently with multi-digit numbers and find common factors and multiples.*

Standard: **6.NS.3**

Fluently (efficiently, accurately, and flexibly) add, subtract, multiply, and divide multi-digit decimals using an efficient algorithm for each operation. **(6.NS.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.NS.2

Explanations and Examples:

Procedural fluency is defined as "skill in carrying out procedures *flexibly, accurately, efficiently and appropriately*". In 4th and 5th grades, students added and subtracted decimals. Multiplication and division of decimals was introduced in 5th grade (to the hundredth place). This standards specifies that students are to be using algorithms that are **efficient** – the traditional standard algorithm is not always the most efficient.

At the elementary level, these operations were based on concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. In 6th grade, students become fluent in the use of multiple efficient algorithms with all operations.

Standard 6.NS.3 calls for students to fluently compute with decimals. A companion of fluency is the extension of the students' existing number sense with decimals. It is insufficient to merely teach procedures about "where to move the decimal." Rather, the focus of instruction and student work should be on operations and number sense.

The use of estimation strategies supports student understanding of operating on decimals.

Example:

First, students estimate the sum and then find the exact sum of 14.4 and 8.75. An estimate of the sum might be 14 + 9 or 23. Students may also state if their estimate is low or high. They would expect their answer to be greater than 23. They can use their estimates to self-correct.

Answers of 10.19 or 101.9 indicate that students are not considering the concept of place value when adding (adding tenths to tenths or hundredths to hundredths) whereas answers like 22.125 or 22.79 indicate that students are having difficulty understanding how the four-tenths and seventy- five hundredths fit together to make one whole and 25 hundredths.

Students use the understanding they developed in 5th grade related to the patterns involved when multiplying and dividing by powers of ten to develop fluency with operations with multi-digit decimals.

Instructional Strategies: See 6.NS.2

Explanations and Examples:
Many of these tasks are not examples of asking students to compute using the traditional standard algorithm for multiplication and division because most educators are familiar with that type of problems. Instead, these tasks show reasoning and estimation strategies students need to develop in order to support their algorithmic computations.

Use the fact that $13 \times 17 = 221$ to find the following. How does knowing the answer to the original problem help you?

a) 13×1.7
b) 130×17
c) 13×1700
d) 1.3×1.7
e) 2210×13
f) $2210 \div 13$
g) $22100 \div 17$
h) $221 \div 1.3$

All these solutions use the associative and commutative properties of multiplication (explicitly or implicitly).

a) $13 \times 1.7 = 13 \times (17 \times 0.1) = (13 \times 17) \times 0.1$, so the product is one-tenth the product of 13 and 17. In other words, $13 \times 1.7 = 22.1$
b) Since one of the factors is ten times one of the factors in 13×17, the product will be ten times as large as well: $130 \times 17 = 2210$
c) $130 \times 1700 = 13 \times (10 \times 100) = (13 \times 17) \times 100, so\ 13 \times 1700 = 22100$
d) Since each of the factors is one tenth the corresponding factor in 13×17, the product will be one one-hundredth as large: $1.3 \times 1.7 = 2.21$
e) $2210 \div 13 = ?$ is equivalent to $13 \times ? = 2210$. Since the product is ten times as big and one of the factors is the same, the other factor must be ten times as big. So $2210 \div 13 = 170$.
f) As in the previous problem, the product is 100 times as big, and since one factor is the same, the other factor must be 100 times as big: $22100 \div 17 = 1300$.
g) $221 \div 1.3 = ?$.is equivalent to $1.3 \times ? = 221$. Since the product is the same size and one of the factors is one-tenth the size, the other factor must be ten times as big. So $221 \div 1.3 = 170$.

Place a decimal in the number on the right side of the equal sign to make the equation true.
Explain your reasoning for each.

1. $3.58 \times 1.25 = 044750$
2. $26.97 \div 6.2 = 04350$

Solution: Reasoning from the meanings of division and multiplication

1. $3.58 \times 1.25 = 4.475$. We are multiplying a number about half-way between 3 and 4 by a number a little more than 1. More specifically, we can appeal to the meaning of multiplication and ask, "How many 3.58's do we have?" A little more than one of them. Thus, the product must be a number around 4. We can also say that the product must be greater than 3 × 1 = 3 and less than 4 × 1.5 = 6. Assuming the digits in the number shown to the right of the equal sign are correct, the only place that would make sense to put the decimal, which would result in a value between 3 and 6, would be 4.475.
2. $26.97 \div 6.2 = 4.35$. We are dividing a number around 27 by a number a little more than 6. More specifically, we can appeal to the meaning of division and ask, "How many 6.2's go into 26.97?" Since 4 sixes go into 24, and 5 sixes go into 30, it is reasonable for the quotient to be a number around 4.5. So the decimal would be placed between the 4 and the 3 to make 4.35.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.B.3
  - Reasoning about Multiplication & Division Place Value Part 1
  - Reasoning about Multiplication & Division Place Value Part 2
  - Pennies to Heaven
  - Buying Gas
  - Gifts from Grandma: Variation 3
  - Movie Tickets
  - Jayden's Snacks
  - Setting Goals
  - 2 Units Wide and 3 Units Long
  - Tenths of Tenths and Hundredths of Hundredths
  - 12 Rectangular Units
  - Adding Base 10 Numbers, Part 1
  - Adding Base 10 Numbers, Part 2
  - Adding Base 10 Numbers, Part 3
  - What is the Best Way to Divide
  - Changing Currency

Open Middle:

- Decimal Addition
- Adding Decimals 2
- Adding Decimals to Make Them As Close to One as Possible

Open Middle
Challenging math problems worth solving

Domain: The Number System (NS)

● **Cluster B:** *Compute fluently with multi-digit numbers and find common factors and multiples.*

Standard: **6.NS.4**

Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express* $18 + 48\ as\ 6(3 + 8)$. **(6.NS.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.MP.6

Connections: See 6.NS.2

Explanations and Examples:

Greatest common factor (GCF) and *least common multiple (LCM)* are usually taught as a means of combining fractions with unlike denominators. This cluster builds upon the previous learning of the **multiplicative structure** of whole numbers, as well as *prime and composite numbers in Grade 4*. Although the process is the same, the point is to become aware of the **relationships** between numbers and their multiples. For example, consider answering the question: "If two numbers are multiples of four, will the sum of the two numbers also be a multiple of four?" Being able to see and write the relationships between numbers will be beneficial as further algebraic understandings are developed. Another focus is to be able to see how the **GCF** is useful in expressing the numbers using the distributive property, (36 + 24) = 12(3+2), where 12 is the **GCF** of 36 and 24. This concept will be extended in the domain Expressions and Equations as work progresses from understanding the number system and solving equations to rewriting and solving algebraic equations in Grade 7.

Note on simplifying: the standards do indeed quite consciously avoid the word "simplify", the point being that different forms of expressions are useful for different purposes, and there is often no mathematical reason to call one of those forms the simplest. This is in accord with MP7, Look for and make use of structure. Students are expected to be able to make strategic choices about what manipulation they perform for the purpose at hand, rather than respond mechanically to commands like "simplify" (McCallum 2012).

Students will find the greatest common factor (GCF) of two whole numbers less than or equal to 100. For example, the GCF of 40 and 16 can be found by:

1) Listing the factors of 40 (1, 2, 4, 5, 8, 10, 20, 40) and 16 (1, 2, 4, 8, 16), then taking the greatest common factor (8) of both sets of factors. Eight (8) is also the largest number such that the other factors are **relatively prime** (two numbers with no common factors other than one). For example, 8 would be multiplied by 5 to get 40; 8 would

be multiplied by 2 to get 16. Since the 5 and 2 are relatively prime, then 8 is the greatest common factor. If students think 4 is the greatest, then show that 4 would be multiplied by 10 to get 40, while 16 would be 4 times 4. Since the 10 and 4 are not relatively prime (have 2 in common), the 4 cannot be the greatest common factor.

Often teachers jump into the practice of having students list the prime factors when finding the GCF however, this is not an expectation of the standards. Greatest common factors and least common multiples are treated with a very light touch in the standards. They are not a major topic, and limited to numbers less than or equal to 100 (6.NS.4). For such numbers, listing the factors or multiplies is probably the most efficient method, and has the added benefit of reinforcing number facts. It also supports the meaning of the terms: you can see directly that you are finding the greatest common factor or the least common multiple. The prime factorization method can be a bit mysterious in this regard … Achieving the focus of the standards means giving some things up, and this is one of those things. (McCallum, 2012).

Students also understand that the greatest common factor of two prime numbers will be 1.

Students use the greatest common factor and the distributive property to find the sum of two whole numbers. For example, 36 + 8 can be expressed as $4(9 + 2(0)) = 4(11)$.

Students find the least common multiple of two whole numbers less than or equal to twelve. For example, the least common multiple of 6 and 8 can be found by:

1) listing the multiplies of 6 (6, 12, 18, 24, 30, …) and 8 (8, 26, 24, 32, 40…), then taking the least in common from the list (24); or

Example

What is the greatest common factor (GCF) of 24 and 36? How can you use factor lists or the prime factorizations to find the GCF?

Solution: $2^2 \bullet 3 = 12$. Students should be able to explain that both 24 and 36 have 2 factors of 2 and one factor of 3, thus 2 x 2 x 3 is the greatest common factor.)

Example

What is the least common multiple (LCM) of 12 and 8? How can you use multiple lists or the prime factorizations to find the LCM?

Solution: $2^3 \cdot 3 = 24$. Students should be able to explain that the least common multiple is the smallest number that is a multiple of 12 and a multiple of 8. To be a multiple of 12, a number must have 2 factors of 2 and one factor of 3(2 x 2 x 3). To be a multiple of 8, a number must have 3 factors of 2 (2 x 2 x 2). Thus the least common multiple of 12 and 8 must have 3 factors of 2 and one factor of 3 (2 x 2 x 2 x 3).

Example

Rewrite 84 + 28 by using the distributive property. Have you divided by the largest common factor? How do you know?

Given various pairs of addends using whole numbers from 1-100, students should be able to identify if the two numbers have a common factor. If they do, they identify the common factor and use the distributive property to rewrite the expression. They prove that they are correct by rewriting both expressions.

- 27 + 36 = 9(3 + 4)
  63 = 9 x 7
  63 = 63
- 31 + 80

There are no common factors. I know that because 31 is a prime number, it only has 2 factors, 1 and 31. I know that 31 is not a factor of 80 because 2 x 31 is 62 and 3 x 31 is 93.

Instructional Strategies:

Greatest common factor and *least common multiple* are usually taught as a means of combining fractions with unlike denominators. This cluster builds upon the previous learning of the multiplicative structure of whole numbers, as well as prime and composite numbers in Grade 4.

Although the process is the same, the point is to become aware of the relationships between numbers and their multiples. For example, consider answering the question: “If two numbers are multiples of four, will the sum of the two numbers also be a multiple of four?” Being able to see and write the relationships between numbers will be beneficial as further algebraic understandings are developed. Another focus is to be able to see how the GCF is useful in expressing the numbers using the distributive property, (36 + 24) = 12(3+2), where 12 is the GCF of 36 and 24.

Students often confuse the concepts of factors and multiples. One effective way to avoid this confusion is to consistently use the vocabulary of factors and multiples each and every time students work on multiplication and division (i.e. the numbers being multiplied are the factors; the product is the multiple).

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.B.4
    - Factors & Common Factors
    - Multiples & Common Multiples
    - Bake Sale
    - The Florist Shop
    - Adding Multiples

Mathematics Assessment Resource Service:

- Finding Factors and Multiples

Nrich Mathematics:

- Factors and Multiples Resources
- Factors and Multiples Resources - MORE
- Counting Factors
- Factors and Multiples Game
- Factors and Multiples

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- The Factor Game
- Factorize

Open Middle:

Least Common Multiple

Common Misconceptions: See 6.NS.2

Domain: The Number System (NS)

▶ **Cluster C:** *Apply and extend previous understandings of numbers to the system of rational numbers*.

Standard: **6.NS.5**

Understand positive and negative numbers to describe quantities having opposite directions or values *(e.g. temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge);* **(6.NS.5)**

6.NS.5a. Use positive and negative numbers to represent quantities in real-world contexts, **(6.NS.5)**

6.NS.5b. Explaining the meaning of 0 in each situation. **(6.NS.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is the foundation for working with rational numbers, algebraic expressions and equations, functions, and the coordinate plane in subsequent grades.

Explanations and Examples:

Students use rational numbers (fractions, decimals, and integers) to represent real-world contexts and understand the meaning of 0 in each situation. For example, 25 feet *below* sea level can be represented as -25, and 25 feet *above* sea level can be represented as +25. In this scenario, zero would represent sea level.

Instructional Strategies:

The purpose of this cluster (6.NS.5 through 6.NS.8) is to begin study of negative numbers, their relationship to positive numbers, and the meaning and uses of absolute value. Starting with examples of having/owing and above/below zero sets the stage for understanding that there is a mathematical way to describe opposites. Students should already be familiar with the counting numbers (positive whole numbers and zero), as well as with fractions and decimals (also positive), so this is the next step in expanding the number system.

Students need to build an understanding that all numbers have an opposite. These special numbers can be shown on vertical or horizontal number lines, which then can be used to solve simple problems. Demonstration of understanding of positives and negatives involves translating among words, numbers and models:

- Given the words "7 degrees below zero, students are able to show it on a model, such as a thermometer, and write the number (-7).
- Given -4 on a number line, students are able to write a real-life example and mathematically show -4.
- Number lines give the opportunity to model absolute value as the distance from zero.
- Simple comparisons can be made between all numbers and order can be determined. Order can also be established and written mathematically; examples could be -3° C > -5° C or -5° C < -3° C.
- Finally, absolute values should be used to relate contextual problems to their meanings and solutions.

Using number lines to model negative numbers, proves the distance between opposites, and shows understanding of the meaning of absolute value. This understanding easily transfers to the creation and usage of four-quadrant coordinate grids. Points should now be **plotted in all four quadrants of a coordinate grid**. Differences between numbers can be found by counting the distance between numbers on the grid. *Actual computation with negatives and positives is handled in Grade 7*.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.C.5
  - Mile High
  - It's Warmer in Miami

Nrich Mathematics:

- Negative Numbers
- Sea Level
- Positive about Negative Numbers

Estimation 180:

- Integers

Domain: The Number System (NS)

▶ **Cluster C:** *Apply and extend previous understandings of numbers to the system of rational numbers.*

Standard: **6.NS.6**

Understand a rational number as a point on the number line and a coordinate pair as a location on a coordinate plane.

6.NS.6a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, *(e.g.* $-(-3) = 3$*)*, and that 0 is its own opposite. **(6.NS.6a & b)**

6.NS.6b. Recognize signs of numbers in ordered pairs indicate locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. **(6.NS.6c)**

6.NS.6c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane. **(6.NS.6d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is the foundation for working with rational numbers, algebraic expressions and equations, functions, and the coordinate plane in subsequent grades.

Explanations and Examples:

In elementary school, students worked with positive whole numbers, fractions, and decimals on the number line. In 6th grade, students extend the number line to represent all rational numbers and recognize that number lines may be either horizontal or vertical (such as a thermometer).

Students recognize that a number and its opposite are equidistance from zero (reflections with zero as the center point). The opposite sign (–) shifts the number to the opposite side of 0. For example, – 4 could be read as "the opposite of 4" which would be negative 4. The following example, – (–6.4) would be read as "the opposite of the opposite of 6.4" which would be 6.4.

Students worked with Quadrant I in elementary school. As the *x*-axis and y-axis are extending to include negatives, students begin to work with the Cartesian coordinate system. Students should recognize the point where the *x*-axis and *y*-axis intersect as the **origin**. Students identify the four quadrants and are able to identify the quadrant for an ordered pair based on the signs of the coordinates. For example, students recognize that in Quadrant II, the signs of all ordered pairs would be (–, +).

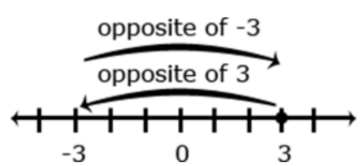
Explanations and Examples: 6.NS.6

Students understand the relationship between two ordered pairs differing only by signs as reflections across one or both axes. For example, in the ordered pairs (-2, 4) and (-2, -4), the *y*-coordinates differ only by signs, which represents a reflection across the *x*-axis. A change is the *x*-coordinates from (-2, 4) to (2, 4), represents a reflection across the *y*-axis. When the signs of both coordinates change, [(2, -4) changes to (-2, 4)], the ordered pair has been reflected across both axes.

Students are able to plot all rational numbers on a number line (either vertical or horizontal) and identify the values of given points on a number line. For example, students are able to identify where the following numbers would be on a number line:

$$-4.5, \qquad 2, \qquad 3.2, \qquad -3, \qquad 3\frac{3}{5}, \qquad 0.2, \qquad -2, \frac{11}{2}$$

Number lines can be used to show numbers and their opposites. Both 3 and -3 are 3 units from zero on the number line. Graphing points and reflecting across zero on a number line extends to graphing and reflecting points across axes on a coordinate grid. The use of both horizontal and vertical number line models facilitates the movement from number lines to coordinate grids.



Example Task:

Graph the following points in the correct quadrant of the coordinate plane. If you reflected each point across the x-axis, what are the coordinates of the reflected points? What similarities do you notice between coordinates of the original point and the reflected point?

$$\left(\frac{1}{2}, -3\frac{1}{2}\right) \qquad \left(-\frac{1}{2}, -3\right) \qquad (0.25, -0.75)$$

Instructional Strategies: See 6.NS.5

Resources/Tools:

See engageNY Module 3

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.C.6
  - Extending the Number Line
  - Location in the Coordinate Plane
- 6.NS.C.6.a
  - Integers on the Number Line 2
- 6.NS.C.6.b
  - Reflecting Points over Coordinate Axes
- 6.NS.C.6.c
  - Plotting Points in the Coordinate Plane

Estimation 180:

- Fun with a Dot and a Line

Common Misconceptions:

Generally, negative values are introduced with integers instead of with fractions and decimals. However, it is a mistake to stop with integers values because students must understand where numbers like –4.5 and $1\frac{3}{4}$ belong in relation to the integers. Students often place $1\frac{3}{4}$ between –1 and 0 instead of between –2 and –1.

Domain: The Number System (NS)

▶ **Cluster C:** *Apply and extend previous understandings of numbers to the system of rational numbers*.

Standard: **6.NS.7**

Understand ordering and absolute value of rational numbers.

6.NS.7a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret* $-3 > -7$ *as a statement that –3 is located to the right of –7 on a number line oriented from left to right.* **(6.NS.7a)**

6.NS.7b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write* $-3°C > -7°C$ *to express the fact that* $-3°C$ *is warmer than* $-7°C$. **(6.NS.7b)**

6.NS.7c. Explain the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of –30 dollars, write* $|-30| = 30$ *to describe the size of the debt in dollars.* **(6.NS.7c)**

6.NS.7d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.* **(6.NS.7d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.NS.5

Explanations and Examples:

a. Students identify the absolute value of a number as the distance from zero but understand that although the value of -7 is less than -3, the absolute value (distance) of -7 is greater than the absolute value (distance) of -3. Students use inequalities to express the relationship between two rational numbers, understanding that the value of numbers is smaller moving to the left on a number line. For example,- $4\frac{1}{2} < -2$ because - $4\frac{1}{2}$ is located to the left of –2 on the number line.

b. Students write statements using < or > to compare rational number in context. However, explanations should reference the context rather than "less than" or "greater than". For example, the balance in Sue's checkbook was –12.55. The balance in Ron's checkbook was –10.45. Since –12.55 < –10.45, Sue owes more than Ron. The interpretation could also be "Ron owes less than Sue".

c. Students understand absolute value as the distance from zero and recognize the symbols | | as representing absolute value. For example, $|-7|$ can be interpreted as the distance –7 is from 0 which would be 7. Likewise $|7|$ can be interpreted as the distance that 7 is from 0 which would also be 7. In real-world contexts, the absolute value can be used to describe size or magnitude. For example, for an ocean depth of –900 feet, write $|-900| = 900$ | to describe the distance below sea level.

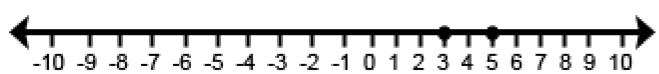
d. When working with positive numbers, the absolute value (distance from zero) of the number and the value of the number is the same; therefore, ordering is not problematic. However, negative numbers have a distinction that students need to understand. As the digit in the negative number increases (moves to the left on a number line), the value of the number decreases. For example, –24 is less than –14 because –24 is located to the left of –14 on the number line. However, absolute value is the distance from zero. In terms of absolute value (or distance) the absolute value of –24 is greater than –14. For negative numbers, as the absolute value increases, the value of the number decreases.

Common models to represent and compare integers include number line models, temperature models and the profit-loss model. On a number line model, the number is represented by an arrow drawn from zero to the location of the number on the number line; the absolute value is the length of this arrow. The number line can also be viewed as a thermometer where each point of on the number line is a specific temperature. In the profit-loss model, a positive number corresponds to profit and the negative number corresponds to a loss. Each of these models is useful for examining values but can also be used in later grades when students begin to perform operations on integers.

In working with number line models, students internalize the order of the numbers; larger numbers on the right or top of the number line and smaller numbers to the left or bottom of the number line. They use the order to correctly locate integers and other rational numbers on the number line.

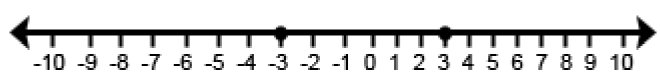
By placing two numbers on the same number line, they are able to write inequalities and make statements about the relationships between the numbers.

Case 1: Two positive numbers



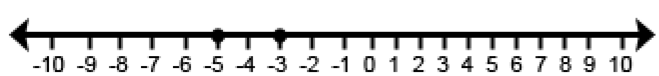
5 > 3
5 is greater than 3

Case 2: One positive and one negative number



3 > -3
positive 3 is greater than negative 3
negative 3 is less than positive 3

Case 3: Two negative numbers



-3 > -5
negative 3 is greater than negative 5
negative 5 is less than negative 3

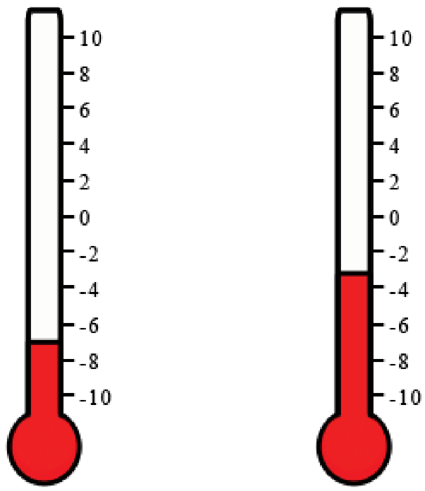
Comparative statements generate informal experience with operations and lay the foundation for formal work with operations on integers in grade 7.

Example:

Write a statement to compare –4 and –2. Explain your answer.

Sample Responses: $-4 < -2$, because –4 is located to the left of –2 on the number line and is a greater distance from zero therefore –2 is greater than –4

One of the thermometers below shows -3°C and the other shows -7°C.
Which thermometer shows which temperature?
Which is the colder temperature? How much colder?
Write an inequality to show the relationship between the temperatures and explain how the model shows this relationship.



Solution:
The thermometer on the left shows -7°C and the one on the right shows -3°C
-7°C is the colder temperature by 4 degrees.
-7°C < -3°C or -3°C > -7°C
The levels on the thermometers show the distance each is from zero and the difference between the two distances.

Students recognize the distance from zero as the absolute value or magnitude of a rational number. Students need multiple experiences to understand the relationships between numbers, absolute value, and statements about order.

Example:

The Great Barrier Reef is the world's largest reef system and is located off the coast of Australia. It reaches from the surface of the ocean to a depth of 150 meters. Students could represent this value as less than 150 meters or a depth no greater than 150 meters below sea level.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.C.7
  - Jumping Flea
  - Above and below sea level
- 6.NS.C.7.a
  - Fractions on the Number Line
  - Integers on the Number Line
- 6.NS.C.7.b
  - Comparing Temperatures

Mathalicious:

- Downside Up

Common Misconceptions: See 6.NS.6

Domain: The Number System (NS)

▶ **Cluster C:** *Apply and extend previous understandings of numbers to the system of rational numbers.*

Standard: **6.NS.8**

Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate. **(6.NS.8)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.

Connections: See 6.NS.5; 6.G.3

Explanations and Examples: 6.NS.8

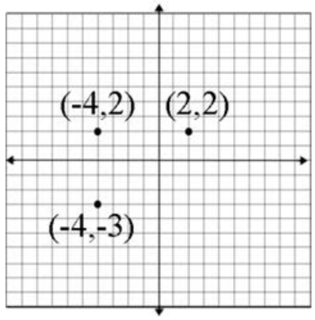
Students find the distance between points whose ordered pairs have the same x-coordinate (vertical) or same y-coordinate (horizontal). For example, the distance between (–5, 2) and (–9, 2) would be 4 units.

This would be a horizontal line since the *y*-coordinates are the same. In this scenario, both coordinates are in the same quadrant. The distance can be found by using a number line to find the distance between –5 and –9. Students could also recognize that –5 is 5 units from 0 (absolute value) and that –9 is 9 units from 0 (absolute value). Since both of these are in the same quadrant, the distance can be found by finding the difference between 9 and 5. $(|9| - |5|)$.

Coordinates could also be in two quadrants. For example, the distance between (3, –5) and (3, 7) would be 12 units. This would be a vertical line since the *x*-coordinates are the same. The distance can be found by using a number line to count from –5 to 7 or by recognizing that the distance (absolute value) from –5 to 0 is 5 units and the distance (absolute value) from 0 to 7 is 7 units so the total distance would be 5 + 7 or 12 units.

Example:
If the points on the coordinate plane below are the three vertices of a rectangle, what are the coordinates of the fourth vertex? How do you know? What are the length and width of the rectangle?

To determine the distance along the *x*-axis between the point (-4, 2) and (2, 2) a student must recognize that -4 is $|-4|$ or 4 units to the left of 0 and 2 is $|2|$ or 2 units to the right of zero, so the two points are total of 6 units



Resources/Tools:
Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.NS.C.8
    - Distances between Points
    - Nome, Alaska

For detailed information, see Learning Progressions, Number System

Common Misconceptions: See 6.NS.6

Domain: Expressions and Equations (EE)

▶ **Cluster A**: *Apply and extend previous understanding of arithmetic to algebraic expressions*.

Standard: **6.EE.1**

Write and evaluate numerical expressions involving whole-number exponents. **(6.EE.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: **6.EE.1 through 6.EE.4**

This cluster is connected to:

- Grade 6 - **Writing, interpreting and using expressions, and equations**.
- The learning in this cluster is foundational in the transition to algebraic representation and problem solving which is extended and formalized in Grade 7, the Number System and Expressions and Equations.

Explanations and Examples: 6.EE.1

Students demonstrate the meaning of exponents to write and evaluate numerical expressions with whole number exponents. The **base** (or **factor**) can be a whole number, positive decimal or a positive fraction.

For example, $\left(\frac{1}{2}\right)^4$ can be written $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, which has the same value as $\frac{1}{16}$.

Students will need to be able to recognize that an expression with a variable represents the same mathematics (i.e., $x^4$4 can be written as $x \cdot x \cdot x \cdot x$) and write algebraic expressions from verbal expressions.

Examples:

Write the following as a numerical expressions using exponential notation.

- The area of a square with a side length of 8 m. (*Solution:* $8^2 m^2$)
- The volume of a cube with a side length of 5 ft. (*Solution*: $5^3 ft^3$)
- Yu-Lee has a pair of mice. The mice each have 2 babies. The babies grow up and have two babies of their own. (*Solution*: 2^3)

Evaluate:

- 4^3 (*Solution*: 64)
- $5 + 2^4 \bullet 6$ (*Solution*: 101)

Instructional Strategies: 6.EE.1

The skills of reading, writing and evaluating expressions are essential for future work with expressions and equations, and are a critical area of focus for Grade 6. In earlier grades, students used grouping symbols ( ) to reduce ambiguity when solving equations. Now the focus is on using ( ) to denote terms in an expression or equation. Likewise, the division symbol (such as, 3 ÷ 5) was used and should now be replaced with a fraction bar $\left(\frac{3}{5}\right)$. Less confusion will occur as students write algebraic expressions and equations if *x* represents only variables and not multiplication. The use of a dot (•) or parentheses between numbers is preferred.

Provide opportunities for students to write expressions for numerical and real-world situations. Write multiple statements that represent a given algebraic expression. For example, the expression $x - 10$ could be written as "ten less than a number," "a number minus ten," "the temperature fell ten degrees," "I scored ten fewer points than my brother," etc. **Students should be able to read an algebraic expression and write a statement**.

Provide opportunities for students to write equivalent expressions, both numerically and with variables. For example, given the expression $x + x + x + x + 4 \bullet 2$, students could write $2x + 2x + 8$ or some other equivalent expression. Make the connection to the simplest form of this expression as $4x + 8$. Because this is a foundational year for building the bridge between the concrete concepts of arithmetic and the abstract thinking of algebra, using hands-on materials (such as algebra tiles, counters, unifix cubes, the "Hands on Algebra" kits) to help students translate between concrete numerical representations and abstract symbolic representations is critical.

Provide expressions and formulas to students, along with values for the variables so students can evaluate the expression. Evaluate expressions using the order of operations with and without parentheses.

Include whole-number exponents, fractions, decimals, etc. Provide a model that shows step-by-step thinking when rewriting an expression. This demonstrates how two lines of work maintain equivalent algebraic expressions and establishes the need to have a way to review and justify thinking.

Provide a variety of expressions and problem situations for students to practice and deepen their skills. Start with simple expressions to evaluate and move to more complex expressions. Likewise start with simple whole numbers and move to fractions and decimal numbers. The use of negatives and positives should mirror the level of introduction in Grade 6's domain of The Number System; students are developing the concept and not generalizing operation rules.

The use of technology can assist in the exploration of the meaning of expressions. Many calculators will allow you to store a value for a variable and then use the variable in expressions. This enables the student to discover how the calculator deals with expressions like x^2, $5x$, xy, and $2(x + 5)$.

Resources/Tools:
For detailed information see, EE Learning Progression

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.A.1
  - The Djinni's Offer
  - Seven to the What?!?
  - Sierpinski's Carpet
  - Exponent Experimentation 1
  - Exponent Experimentation 2
  - Exponent Experimentation 3

Youtube videos:

- Exponential Notation
- Exponential Notation (Intro)

Common Misconceptions:
Misconceptions when dealing with expressions stem from the misunderstanding/reading of the expression. For example, knowing the operations that are being referenced with notation like, x^3, 4*x*, 3(*x* + 2*y*) is critical. The fact that x^3 means x • x • x, not 3 times *x or x* + *x* + *x*.

Domain: Expressions and Equations (EE)

▶ **Cluster A:** *Apply and extend previous understanding of arithmetic to algebraic expressions*.

Standard: **6.EE.2**

Write, read, and evaluate expressions in which letters stand for numbers.

6.EE.2a. Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as* $5 - y$. **(6.EE.2a)**

6.EE.2b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression* $2(8 + 7)$ *as a product of two factors; view* $(8 + 7)$ *as both a single entity and a sum of two terms.* **(6.EE.2b)**

6.EE.2c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas* $V = s^3$ *and* $A = 6s^2$ *to find the volume and surface area of a cube with sides of length* $s = \frac{1}{2}$. **(6.EE.2c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
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- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.EE.1

Explanations and Examples: 6.EE.2a-c

Students write expressions from verbal and/or written descriptions using letters and numbers. Students understand order is important in writing subtraction and division problems. Students understand that the expression "5 times any number *n*" could be represented with 5*n* and that a number and letter written together means to multiply.

Students use appropriate mathematical language when writing or reading algebraic expressions. Students can describe expressions such as 3(2 + 6) as the product of two factors with 3 and (2 + 6) as those two factors. The quantity (2 + 6) can be viewed as one factor consisting of two terms.

Students evaluate algebraic expressions, using order of operations as needed.
Given an expression, such as 3x + 2y, find the value of that expression when x is equal to 4 and y is equal to 2.4.
This problem requires students to understand that multiplication is understood when numbers and variables are written together and that the order of operations must be used to evaluate.

$$3 \cdot 4 + 2 \cdot 2.4$$
$$12 + 4.8$$
$$16.8$$

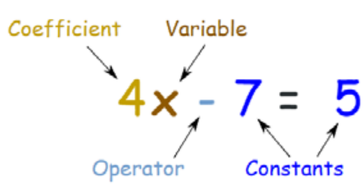
Explanations and Examples: 6.EE.2a through 6.EE.2c
Given a context and the formula arising from the context, students could write an expression and then evaluate for any number.
For example, it costs $100 to rent the skating rink plus $5 per person. The cost for any number (*n*) of people could be found by the expression, 100 + 5*n*. What is the cost for 25 people?

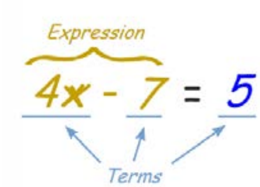
It is important for students to read algebraic expressions in a manner that reinforces that the variable represents a number.

- *r* + 21 as "some number plus 21 as well as "*r* plus 21"
- *n* • 6 as "some number times 6 as well as "*n* times 6"
- $\frac{s}{6}$ and *s*÷ 6 "as some number divided by 6" as well as "*s* divided by 6"

Students should identify the parts of an algebraic expression including variables, coefficients, constants, and the names of operations (sum, difference, product, and quotient). Development of this common language helps students to understand the structure of expressions and explain their process for rewriting expressions.

Variables are letters that represent numbers. There are various possibilities for the numbers they can represent; students can substitute these possible numbers for the letters in the expression for various different purposes.





Visuals from https://www.mathsisfun.com/algebra/definitions.html

A term is either a single number or a variable, or numbers and variables multiplied together. When the term is an explicit number, it is called a constant. When the term is a product of a number and a variable, the number is called the coefficient of the variable.

Consider the following expression: $x^2 + 5y - 3x + 6$

- The variables are *x* and *y*.
- There are 4 terms, x^2, $5y$, $-3x$, and 6.
- There are 3 variable terms, x^2, $5y$, $-3x$. They have coefficients of 1, 5, and -3 respectively. The coefficient of x^2 is 1, since $x^2 = 1x^2$. The term 5y represent 5 *y*'s or 5• y.
- There is one constant term, 6.
- The expression shows a sum of all four terms.

Examples:

- 7 more than 3 times a number $3x + 7$
- 3 times the sum of a number and 5 $3(n + 5)$
- 7 less than the product of 2 and a number $2x - 7$
- Twice the difference between a number and 5 $2(n - 5)$
- Evaluate $5(n + 3) - 7n$, when $n = \frac{1}{2}$.
- The expression $c + 0.07c$ can be used to find the total cost of an item with 7% sales tax, where *c* is the pre-tax cost of the item. Use the expression to find the total cost of an item that cost $25.
- The perimeter of a parallelogram is found using the formula $p = 2l + 2w$. What is the perimeter of a rectangular picture frame with dimensions of 8.5 inches by 11 inches?

Instructional Strategies: See 6.EE.1

Resources/Tools:

From the **National Library of Virtual Manipulatives**: Online algebra tiles that can be used to represent expressions and equations.

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.A.2
  - Rectangle Perimeter 1
  - Distance to School
- 6.EE.A.2.c
  - Families of Triangles

Estimation 180:

- Algebraic Expressions

Youtube videos:

- Algebraic Vocabulary
- Algebraic Expressions

Under the Dome:

- A Math-ic Prediction
- Consecutive Number Sums

MARS:

- Evaluating Statements about Number Operatio
- Evaluating Statements: Consecutive Sums

Robert Kaplinsky's Lessons

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- Algebra Tiles

See also engageNY Modules 4 & 5

Common Misconceptions:

Some students may just substitute the variables with a number and believe they have "solved" the problem. Example: a student may evaluate $4x$ with $x = 7$ using substitution and believe that the answer is 47.

When using the distributive property, students will often multiply the first term, but forget to do the same to the second term.

Students assume if there is not a coefficient in front of a variable, there is not a number there. They do not understand that $y = 1y$.

When solving equations and inequalities, they may use the inverse operation on only one side and on the other or they may use the same operation rather than the inverse.

Domain: Expressions and Equations (EE)

▶ **Cluster A**: *Apply and extend previous understanding of arithmetic to algebraic expressions*

Standard: **6.EE.3**

Apply the properties of operations and combine like terms, with the conventions of algebraic notation, to identify and generate equivalent expressions. *For example, apply the distributive property to the expression* $3(2 + x)$ *to produce the equivalent expression* $6 + 3x$*; apply properties of operations to* $y + y + y$ *to produce the equivalent expression 3y.* **(6.EE.3) (6.EE.4)**

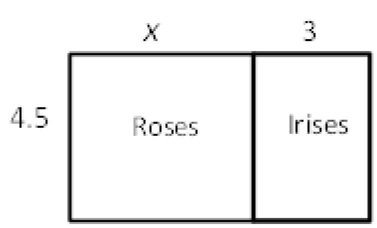
Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

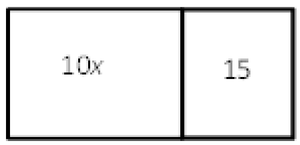
Connections: See 6.EE.1

Explanations and Examples: 6.EE.3

Students need to use and understand the distributive property to write equivalent expressions. For example, area models from elementary are a known model with students that can be used to illustrate the distributive property with variables. Given that the width is 4.5 units and the length can be represented by *x* + 3, the area of the flowers below can be expressed as 4.5(*x* + 3) or 4.5*x* + 13.5.

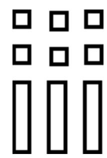


When given an expression representing area, students need to find the factors. For example, the expression 10x +15 can represent the area of the figure below. Students find the greatest common factor (5) to represent the width and then use the distributive property to find the length (2*x* + 3). The factors (dimensions) of this figure would be 5(2*x* + 3).



Students use their understanding of multiplication to interpret *3(2 + x). For example, 3 groups of (2 + x).* They use a model to represent x, and make an array to show the meaning of *3(2 + x).*
They can explain why it makes sense that *3(2 + x)* is equal to *6 + 3x.*

An array with 3 columns and $x + 2$ in each column:



Students interpret *y* as referring to one *y.* Thus, they can reason that one *y* plus one *y* plus one *y* ***must be*** *3y.* They also *use and understand* the distributive property, the multiplicative identity property of 1, and the commutative property for multiplication to prove that $y + y + y = 3y$:

$y + y + y = 1 \bullet y + 1 \bullet y + 1 \bullet y$; so $1 \bullet y + 1 \bullet y + 1 \bullet y = y\,(1 + 1 + 1)$; then $y\,(1 + 1 + 1) = y \bullet 3$; which is $3y$

Students demonstrate an understanding of like terms as quantities being added or subtracted with the same variables and exponents. For example, $3x + 4x$ are like terms and can be combined as $7x$; however, $3x + 4x^2$ are not.

This concept can be illustrated by substituting in a value for *x*. For example, $9x – 3x = 6x$ not 6.
Choosing a value for *x*, such as 2, can prove non-equivalence.

$9(2) – 3(2) = 6(2)$	**however**	$9(2) – 3(2)$? 6
$18 – 6 = 12$		$18 – 6$? 6
$12 = 12$		$12 \neq 6$

Students connect their experiences with finding and identifying equivalent forms of whole numbers and can write expressions in various forms. Students generate equivalent expressions using the associative, commutative, and distributive properties. They can prove that the expressions are equivalent by rewriting each expression into the same form.

Example:

Are the expressions equivalent? How do you know?

4m + 8   *4(m+2)*   *3m + 8 + m*   *2 + 2m + m + 6 + m*

Solution:

Expression	Rewrite the Expression	Explanation
4m + 8	*4m + 8*	Already in simplest form
4(m+2)	*4(m+2)* *4m + 8*	*Distributive property*
3m + 8 + m	*3m + 8 + m* *3m + m + 8* *(3m + m) + 8* *4m + 8*	*Combined like terms*
2 + 2m + m + 6 + m	*2 + 2m + m + 6 + m* *2 + 6 + 2m + m + m* *(2 + 6) + (2m + m + m)* *8 + 4m* *4m + 8*	*Combined like terms*

Instructional Strategies: See 6.EE.1

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.A.3
  - Anna in D.C.
- 6.EE.A.4
  - Rectangle Perimeter 2
  - Equivalent Expressions

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- Pan Balance - Shapes
- Pan Balance – Numbers
- Pan Balance - Expressions

Open Middle
Tasks with Expressions

For detailed information see, EE Learning Progressions

Common Misconceptions: See 6.EE.1

Domain: Expressions and Equations (EE)

Cluster B: *Reason about and solve one-variable equations and inequalities.*

Standard: 6.EE.4

Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true. **(6.EE.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (6.EE.4 through 6.EE.7)

This cluster is connected to:

- Writing, interpreting and using expressions, and equations.

Explanations and Examples: 6.EE.4

Examples:

The equation 0.44*s* = 11 where *s* represents the number of stamps in a booklet. The booklet of stamps costs 11 dollars and each stamp costs 44 cents. How many stamps are in the booklet? Explain the strategies you used to determine your answer. Show that your solution is correct using substitution.

Twelve is less than 3 times another number can be shown by the inequality $12 < 3n$. Which of the following numbers could possibly make this a true statement?

0, 4, 5

Students identify values from a specified set that will make an equation true. For example, given the expression $x + 2\frac{1}{2}$, which of the values below would make $x + 2\frac{1}{2} = 6$ when substituting the *x*? Is there more than one correct answer?

$0, \quad 3\frac{1}{2}, \quad 4$

By using substitution, students identify $3\frac{1}{2}$ as the value that will make both sides of the equation equal.

The solving of inequalities is limited to choosing values from a specified set that would make the inequality true. For example, find the values of x that will make $x + 3.5 \geq 9$.

$$\{5, \quad 5.5, \quad 6, \quad \frac{15}{2}, \quad 10.2, \quad 15\}$$

Using substitution, students identify 5.5, 6, $\frac{15}{2}$, 10.2, and 15 as the values that make the inequality true. NOTE: If the inequality had been $x + 3.5 > 9$, then 5.5 would not work since 9 is not greater than 9.

This standard is foundational to **6.EE.6** and **6.EE.7**

Instructional Strategies: 6.EE.4 through 6.EE.7

In order for students to **understand** equations, the skill of solving an equation must be developed ***conceptually*** before it is developed ***procedurally***. This means that students should be thinking about what numbers could possibly be a solution to the equation before solving the equation. For example, in the equation $x + 21 = 32$ students know that $21 + 9 = 30$ therefore the solution must be 2 more than 9 or 11, so $x = 11$.

Provide multiple situations in which students must determine if a single value is required as a solution, or if the situation allows for multiple solutions. This creates the need for both types of equations (single solution for the situation) and inequalities (multiple solutions for the situation). Solutions to equations should not require using the rules for operations with negative numbers since the conceptual understanding of negatives and positives is being introduced in Grade 6. When working with inequalities, provide situations in which the solution is not limited to the set of positive whole numbers but includes rational numbers. This is a good way to practice fractional numbers and introduce negative numbers. Students need to be aware that numbers less than zero could be part of a solution set for a situation. As an extension to this concept, certain situations may require a solution between two numbers. For example, a problem situation may have a solution that requires more than 10 but not greater than 25. Therefore, the exploration with students as to what this would look like both on a number line and symbolically is a reasonable extension.

The process of translating between mathematical phrases and symbolic notation will also assist students in the writing of equations/inequalities for a situation. This process should go both ways; Students should be able to write a mathematical phrase for an equation. Additionally, the writing of equations from a situation or story does not come naturally for many students. A strategy for assisting with this is to give students an equation and ask them to come up with the situation/story that the equation could be referencing.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.B.5
    - Log Ride
    - Make Use of Structure

Estimation 180:
Woody's Raise

Domain: Expressions and Equations (EE)

▶ **Cluster B:** *Reason about and solve one-variable equations and inequalities.*

Standard: **6.EE.5**

Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. **(6.EE.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.EE.4

Explanations and Examples:

Students write expressions to represent various real-world situations. For example, the expression $a + 3$ could represent Susan's age in three years, when a represents her present age. The expression $2n$ represents the number of wheels on any number of bicycles. Other contexts could include age (Johnny's age in 3 years if a represents his current age) and money (value of any number of quarters).

Given a contextual situation, students define variables and write an expression to represent the situation. For example, the skating rink charges $100 to reserve the place and then $5 per person. Write an expression to represent the cost for any number of people.

N = the number of people

$100 + 5n$

Connecting writing expressions with story problems and/or drawing pictures will give students a context for this work. It is important for students to read algebraic expressions in a manner that reinforces that the variable represents a number.

Examples:

- Maria has three more than twice as many crayons as Elizabeth. Write an algebraic expression to represent the number of crayons that Maria has.
  (*Solution*: $2c + 3$ where c represents the number of crayons that Elizabeth has.)
- An amusement park charges $28 to enter and $0.35 per ticket. Write an algebraic expression to represent the total amount spent.
  (*Solution*: $28 + 0.35t$ where t represents the number of tickets purchased).
- Andrew has a summer job doing yard work. He is paid $15 per hour and a $20 bonus when he completes the yard. Write an expression to represent the amount of money he earned.
  (*Solution*: $15h + 20$ where h is the number of hours worked)
- Describe a problem situation that can be represented using the expression $2c + 3$; where c represents the cost of an item.
- Bill earned $5.00 mowing the lawn on Saturday. He earned more money on Sunday. Write an expression that shows the amount of money Bill has earned.
  (Solution: $5.00 + *n*)

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.B.6
  - Firefighter Allocation
  - Pennies to Heaven

Under the Dome:

- Consecutive Number Sums
- Triangle Mystery

UNDER THE DOME
Thoughts about Teaching

Open Middle:

- Expressions and Equations tasks

Open Middle
Challenging math problems worth solving

Domain: Expressions and Equations (EE)

▶ **Cluster B:** *Reason about and solve one-variable equations and inequalities.*

Standard: **6.EE.6**

Write and solve one-step equations involving non-negative rational numbers using addition, subtraction, multiplication and division. **(6.EE.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP 1 Make sense of problems and persevere in solving them.
- ✓ MP 2 Reason abstractly and quantitatively.
- ✓ MP 3 Construct viable arguments and critique the reasoning of others.
- ✓ MP 4 Model with mathematics.
- ✓ MP 7 Look for and make use of structure.

Connections: See 6.EE.4

Explanations and Examples:

Students have used algebraic expressions to generate answers given values for the variable. This understanding is now expanded to equations where the value of the *variable* is unknown but the *outcome* is known. For example, in the expression, $x + 4$, any value can be substituted for the x to generate a numerical answer; however, in the equation $x + 4 = 6$, there is only one value that can be used to get a 6. **Problems should be in context when possible and use only one variable**.

Students write equations from real-world problems and then use inverse operations to solve one-step equations. Equations may include fractions and decimals with non-negative solutions.

Students create and solve equations that are based on real-world situations. It may be beneficial for students to draw pictures that illustrate the equation in problem situations. Solving equations using reasoning and prior knowledge should be required of students to allow them to develop effective strategies.

Example:

Meagan spent $56.58 on three pairs of jeans. If each pair of jeans costs the same amount, write an algebraic equation that represents this situation and solve to determine how much one pair of jeans cost.

$56.58		
J	J	J

Sample Solution: Students might say: "I created the bar model to show the cost of the three pairs of jeans. Each bar labeled *J* is the same size because each pair of jeans costs the same amount of money. The bar model represents the equation $3J = \$56.58$.

To solve the problem, I need to divide the total cost of 56.58 between the three pairs of jeans. I know that it will be more than $10 each because 10 x 3 is only 30 but less than $20 each because 20 x 3 is 60. If I start with $15 each, I am up to $45. I have $11.58 left. I then give each pair of jeans $3. That's $9 more dollars. I only have $2.58 left. I continue until all the money is divided. I ended up giving each pair of jeans another $0.86. Each pair of jeans costs $18.86 (15+3+0.86).

When the work is double checked showing that the jeans cost $18.86 each, then this is shown to solve the equation correctly because $18.86 x 3 is $56.58."

Example:

- Julio gets paid $20 for babysitting. He spends $1.99 on a package of trading cards and $6.50 on lunch. Write and solve an equation to show how much money Julio has left.
  (*Solution:* 20 = 1.99 + 6.50 + *x*, *x* = $11.51)

20		
1.99	6.50	money left over (m)

Instructional Strategies: See 6.EE.4

Resources/Tools:
See Expressions and Equations Learning progression for Grade 6-8 - EE Learning Progressions

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.B.7
  - Firefighter Allocation
  - Morning Walk
  - Anna in D.C.
  - Fruit Salad

Open Middle:

- Expression and Equations tasks

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- Algebra Tiles
- Pan Balance – Numbers

Thinking Blocks:
Access the Thinking Blocks website and scroll down to the Algebra section to find equations that are solved using the bar method.

Robert Kaplinsky's lessons:

- What Does 2000 Calories Look Like?

Domain: Expressions and Equations (EE)

▶ **Cluster B:** ***Reason about and solve one-variable equations and inequalities.***

Standard: **6.EE.7**

Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams. **(6.EE.8)**

Suggested Standards for Mathematical Practice (MP):

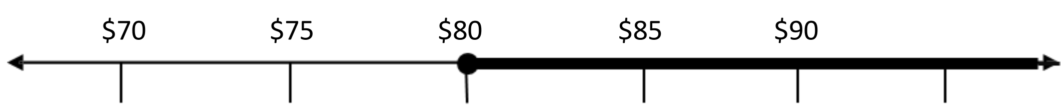
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.EE.4

Explanations and Examples:

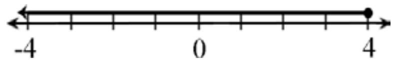
Many real-world situations are represented by inequalities. Students write an inequality and represent solutions on a number line for various contextual situations.

For example, the class must raise at least $80 to go on the field trip. If *m* represents money, then the inequality statement would be *m* ≥ $80. Students should recognize that possible values can include too many decimal values to name. Therefore, the values are represented on a number line by shading.



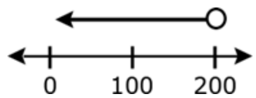
A number line diagram is drawn with an open circle when an inequality contains a < or > symbol to show solutions that are less than or greater than the number but not equal to the number. The circle is shaded, as in the example above, when the number is to be included. Students recognize that possible values can include fractions and decimals, which are represented on the number line by shading. Shading is extended through the arrow on a number line to show that an inequality has an infinite number of solutions.

Examples:

- Graph $x \leq 4$. *Solution:* 

- Jonas spent more than $50 at an amusement park. Write an inequality to represent the amount of money Jonas spent. What are some possible amounts of money Jonas could have spent? Represent the situation on a number line.

    - Less than $200.00 was spent by the Flores family on groceries last month. Write an inequality to represent this amount and graph this inequality on a number line.

      Solution: $200 > x$ 

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.B.8
    - Fishing Adventures 1 (extension which leads to HS inequalities)
    - Height Requirements

Estimation 180:

- Inequalities

Domain: Expressions and Equations (EE)

▶ **Cluster C:** *Represent and analyze quantitative relationships between dependent and independent variables.*

Standard: 6.EE.8

Use variables to represent two quantities in a real-world problem that change in relationship to one another.

- 6.EE.8a. Identify the independent and dependent variable. **(6.EE.9)**
- 6.EE.8b. Write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation* $d = 65t$ *to represent the relationship between distance and time.* **(6.EE.9)**
- 6.EE.8c. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. **(6.EE.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure
- ✓ MP 8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Writing, interpreting and using expressions, and equations.
- It is closely tied to **Ratios and Proportional Relationships**, allowing the ideas in each to be connected and taught together. See **6.RP.3**

Explanations and Examples:

The purpose of this standard is for students to understand the ***relationship*** between two variables, which begins with the distinction between dependent and independent variables. The independent variable is the variable that can be changed; the dependent variable is the variable that is affected by the change in the independent variable. Students recognize that the *independent variable is graphed on the x-axis*; *the dependent variable is graphed on the y-axis*.

Students recognize that not all data should be graphed with a line. Data that is **discrete** would be graphed with coordinates only. *Discrete data is data that would not be represented with fractional parts such as people, tents, records, etc.* For example, a graph illustrating the cost per person would be graphed with points since part of a person would not be considered. A line is drawn when both variables could be represented with fractional parts.

Students are expected to recognize and explain the impact on the dependent variable when the independent variable changes (as the *x* variable increases, students need to figure out how the *y* variable changes). Relationships should be proportional with the line passing through the origin. Additionally, students should be able to write an equation from a word problem and understand how the coefficient of the dependent variable is related to the graph and/or a table of values.

Students can use many forms to represent relationships between quantities. Multiple representations include describing the relationship using language, a table, an equation, or a graph. Translating between multiple representations helps students understand that each form represents the same relationship and provides a different perspective on the function.

Instructional Strategies:

The goal is to help students connect the pieces together. This can be done by having students use multiple representations for the mathematical relationship. Students need to be able to translate freely among the story, words (mathematical phrases), models, tables, graphs and equations. They also need to be able to start with any of the representations and develop the others.

Provide multiple situations for the student to analyze and determine what unknown is dependent on the other components. *For example, how far I travel is dependent on the time and rate that I am traveling*.

Throughout the *Expressions and Equations* domain in Grade 6, students need to build an understanding of how the expressions or equations relate to situations presented, as well as the process of solving those expressions or equations.

The use of technology, including computer apps, CBLs, and other hand-held technology allows the collection of real-time data or the use of actual data to create tables and charts. It is valuable for students to realize that although real-world data often is not linear, a line sometimes can model the data.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.EE.C.9
  - Chocolate Bar Sales
  - Families of Triangles

Common Misconceptions:

Students may misunderstand what the graph represents when placed in context. For example, the line that shows the moving up or down on a graph does not necessarily mean that a person is moving up or down. Discuss what it could mean.

Domain: Geometry (G)

◆ **Cluster A:** *Solve real-world and mathematical problems involving area, surface area, and volume*.

Standard: 6.G.1

Find the area of all triangles, special quadrilaterals (including parallelograms, kites and trapezoids), and polygons whose edges meet at right angles (rectilinear figures (*See Geometry Progression K-6 Pg. 19 Paragraph 4*) *polygons*) by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. **(6.G.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: **6.G.1 through 6.G.4**

This cluster focuses on:

- Additional content for development.
- Students in Grade 6 build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume.
- An understanding of how to find the area, surface area and volume of an object is developed in Grade 5 and should be built upon in Grade 6 to facilitate understanding of the formulas found in Measurement and Data and when to use the appropriate formula.

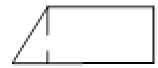
The use of floor plans and composite shapes on dot paper is a foundational concept for scale drawing and determining the actual area based on a scale drawing that will be introduced in Grade 7 (Geometry and Ratio and Proportional Relationships).

Explanations and Examples: 6.G.1

Students continue to understand that **area** is the number of squares needed to cover a plane figure. Finding the area of triangles is introduced in relationship to the area of **rectangles** – a rectangle can be decomposed into two congruent triangles. Therefore, the area of the triangle is ½ the area of the rectangle. The area of a rectangle can be found by multiplying base • height; therefore, the area of the triangle is $\frac{1}{2}bh$ or $\frac{(bh)}{2}$. Students decompose shapes into rectangles and triangles to determine the area.

For example, a trapezoid can be decomposed into triangles and rectangles (see figures below). Using the trapezoid's dimensions, the area of the individual triangle(s) and rectangle can be found and then added together.

Isosceles trapezoid



Right trapezoid

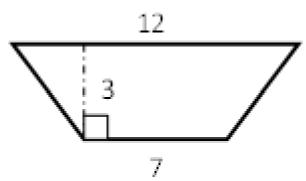
Students should know the formulas for rectangles and triangles. "Knowing the formula" does not mean memorization of the formula. To "know" means to have an understanding of ***why*** the formula works and **how** the formula relates to the measure (area) and the figure.

Special quadrilaterals include rectangles, squares, parallelograms, trapezoids, rhombi, and kites. Students can use tools such as the Isometric Drawing Tool on NCTM's Illuminations site to shift, rotate, color, decompose and view figures in 2D or 3D.

Explanations and Examples: 6.G.1

- Find the area of a triangle with a base length of three units and a height of four units.

- Find the area of the trapezoid shown below using the formulas for rectangles and triangles.

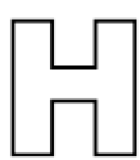


- A rectangle measures 3 inches by 4 inches. If the lengths of each side double, what is the effect on the area?

- The area of the rectangular school garden is 24 square units. The length of the garden is 8 units. What is the length of the fence needed to enclose the entire garden?

- The sixth grade class at Hernandez School is building a giant wooden H for their school. The H will be 10 feet tall and 10 feet wide and the thickness of the block letter will be 2.5 feet.

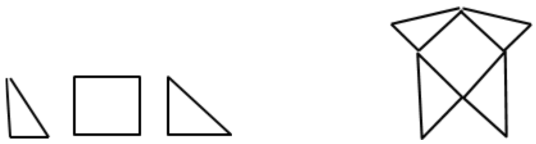
    - How large will the H be if measured in square feet?
    - The truck that will be used to bring the wood from the lumber yard to the school can only hold a piece of wood that is 60 inches by 60 inches.
    - What pieces of wood (how many pieces and what dimensions) are needed to complete the project?



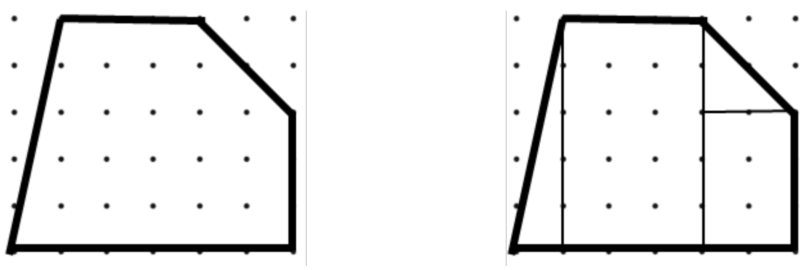
Instructional Strategies: 6.G.1

It is very important for students to continue to physically manipulate materials and make connections to the symbolic and more abstract aspects of geometry. Exploring possible nets should be done by taking apart (unfolding) three-dimensional objects. This process is also foundational for the study of surface area of prisms. Building upon the understanding that a net is the two-dimensional representation of the object, students can apply the concept of area to find surface area. The surface area of a prism is the sum of the areas for each face.

Multiple strategies can be used to aid in the skill of determining the area of simple two-dimensional composite shapes. A beginning strategy should be to use rectangles and triangles, building upon shapes for which they can already determine area to create composite shapes. This process will reinforce the concept that composite shapes are created by joining together other shapes, and that the total area of the two-dimensional composite shape is the sum of the areas of all the parts



A follow-up strategy is to place a composite shape on grid or dot paper. This aids in the decomposition of a shape into its foundational parts. Once the composite shape is decomposed, the area of each part can be determined and the sum of the area of each part is the total area.



Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.G.A.1
  - Same Base and Height, Variation 1
  - Same Base and Height, Variation 2
  - Finding Areas of Polygons
  - Base and Height
  - Polygons in the Coordinate Plane
  - Sierpinski's Carpet
  - Wallpaper Decomposition
  - 24 Unit Squares
  - Areas of Right Triangles
  - Areas of Special Quadrilaterals

MARS:

- Designing: Candy Cartons.
- Candle Box
- Smoothie Box
- Fruit Boxes

Dan Meyer's 3 Act Tasks:

- Bubble Wrap
- Dollar Wall

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- "Area Contractor": Students explore surface area in a real-world application by providing an estimate as a contractor would to a potential customer. Students determine surface area.

Georgia Department of Education:

- "Recarpeting the Classroom ": In a real world problem student work with area of the classroom as they determine the amount of carpeting required.

Estimation 180:
Fun with a Sticky

Illustrative Mathematics:

- Finding Areas of Polygons: Variation 1
- Base and Height
- Same Base and Height: Variation 2

Open Middle:
Volume and Surface Area tasks

Common Misconceptions:

Students may believe that the orientation of a figure changes the figure. In Grade 6, some students still struggle with recognizing common figures in different orientations.
For example, some students view a square rotated 45° as no longer a square and call it a diamond, instead of a square.

This impacts students' ability to decompose composite figures and to appropriately apply formulas for area.

Providing multiple orientations of objects within classroom examples and work is essential for students to overcome this misconception.

Students often forget or confuse the formulas for area, surface area and volume. Exposing students to these concepts in a manner in which they understand the meaning behind the terms as the standards suggest will be important in fostering their conceptual development.

Care should be given to ensuring students understand the units that each of these terms require:

- Perimeter - linear units (*cm, m, in, yd*)
- area and surface area - square units(*sq. cm, sq. m, sq. in, sq. yd*) 
- volume- cubic units (cm^3) 

Domain: Geometry (G)

◆ **Cluster A:** *Solve real-world and mathematical problems involving area, surface area, and volume.*

Standard: 6.G.2

Find the volume of a right rectangular prism with fractional edge lengths by applying the formulas $V = lwh \text{ and } V = Bh$ (*B* is the area of the base and *h* is the height) to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems. (Builds on the 5th grade concept of packing unit cubes to find the volume of a rectangular prism with whole number edge lengths.) **(6.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.G.1

Explanations and Examples: 6.G.2

Previously students calculated the volume of right rectangular prisms (boxes) using whole number edges. The unit cube was 1 x 1 x 1. In 6th grade the unit cube will have fractional edge lengths. (i.e., $\frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$) Students find the volume of the right rectangular prism with these unit cubes.

For example, if a right rectangular prism has edges of $1\frac{1}{4}$”, 1” and $1\frac{1}{2}$”. The volume can be found by recognizing that the unit cube would be $\frac{1}{4}$” on all edges, changing the dimensions to $\frac{5}{4}$”, $\frac{4}{4}$”, and $\frac{6}{4}$”. The volume is the number of unit cubes making up the prism (5 x 4 x 6), which is 120 unit cubes each with a volume of $\frac{1}{64}$ ($\frac{1}{4}$” • $\frac{1}{4}$” • $\frac{1}{4}$”). This can also be expressed as $\frac{5}{4}$” • $\frac{4}{4}$” • $\frac{6}{4}$” OR $\frac{120}{64}$”

“*Know the formula*” does not mean memorization of the formula. To “know” means to have an understanding of ***why*** the formula works and **how** the formula relates to the measure (volume) and the figure.

Students need multiple opportunities to measure volume by filling rectangular prisms with blocks and looking at the relationship between the total volume and the area of the base. Through these experiences, students derive the volume formula (volume equals the area of the base times the height). Students can explore the connection between filling a box with unit cubes and the volume formula using interactive applets such as the Cubes Tool on NCTM’s Illuminations.

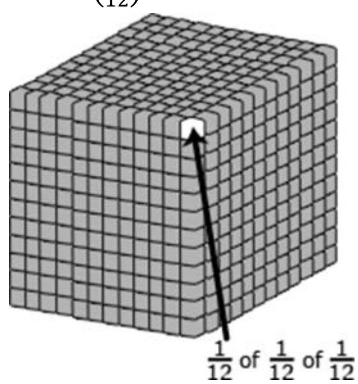
In addition to filling boxes, students can draw diagrams to represent fractional side lengths, connecting with multiplication of fractions. This process is similar to composing and decomposing two dimensional shapes.

Fill prisms with cubes of different edge lengths (including fractional lengths) to explore the relationship between the length of the repeated measure and the number of units needed.

An essential understanding to this strategy is that the volume of a rectangular prism does not change when the units used to measure the volume changes, just the units to describe the volume changes but the amount of actual space remains the same. Since a focus in Grade 6 is to use fractional lengths to measure, if the same object is measured using one centimeter cubes and then measured using half centimeter cubes, the volume will appear to be eight times greater with the smaller unit. However, students need to understand that the value or the number of cubes is greater but the volume is the same.

Explanations and Examples: 6.G.2

- The model shows a cubic foot filled with cubic inches. The cubic inches can also be labeled as a fractional cubic unit with dimensions of $\frac{1}{12}\ ft$ and volume of $\left(\frac{1}{12}\right)^3 ft^3$.



Instructional Strategies: 6.G.2

Fill prisms with cubes of different edge lengths (including fractional lengths) to explore the relationship between the length of the repeated measure and the number of units needed.

An essential understanding to this strategy is the volume of a rectangular prism does not change when the units used to measure the volume changes. Since focus in Grade 6 is to use fractional lengths to measure, if the same object is measured using one centimeter cubes and then measured using half centimeter cubes, the volume will appear to be eight times greater with the smaller unit.

However, students need to understand that the value or the number of cubes is greater but the volume is the same.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.G.A.2
  - Computing Volume Progression 1
  - Computing Volume Progression 4
  - Banana Bread
  - Computing Volume Progression 2
  - Computing Volume Progression 3
  - Volumes with Fractional Edge Lengths

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- Cubes
- Isometric Drawing Tool

MARS:

- Designing 3D Products: Candy Cartons
- Using Space Efficiently: Packing a Truck

Robert Kaplinsky's Lessons:

- How Much Money is That?

Open Middle:

- Volume and Surface Area tasks

Nrich Mathematics:

- Making Boxes
- Surface Area and Volume Tasks
- Cuboid Challenge
- Boxed In

Common Misconceptions: See 6.G.1

Domain: Geometry (G)

◆ **Cluster A:** *Solve real-world and mathematical problems involving area, surface area, and volume.*

Standard: 6.G.3

Draw polygons whose edges meet at right angles (rectilinear figure polygons) in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems. **(6.G.3)**.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.G.1 & 6.NS.8

Explanations and Examples: 6.G.3

Students should be provided many experiences with the coordinate plane. They are to be given coordinates of polygons to draw in the coordinate plane. If both *x*-coordinates are the same (2, -1) and (2, 4), then students recognize that a vertical line has been created and the distance between these coordinates is the distance between -1 and 4, or 5. If both the *y*-coordinates are the same (-5, 4) and (2, 4), then students recognize that a horizontal line has been created and the distance between these coordinates is the distance between -5 and 2, or 7. Using this understanding, student solve real-world and mathematical problems, including finding the area of quadrilaterals and triangles.

This standard can be taught in conjunction with **6.G.1** to help students develop the formula for the triangle by using the squares of the coordinate grid. Given a triangle, students can make the corresponding square or rectangle and realize the triangle is $\frac{1}{2}$ of the area of the rectangle.

Students move from counting the squares to making a rectangle, and recognizing the triangle as $\frac{1}{2}$ leading to the development of the formula for the area of a triangle.

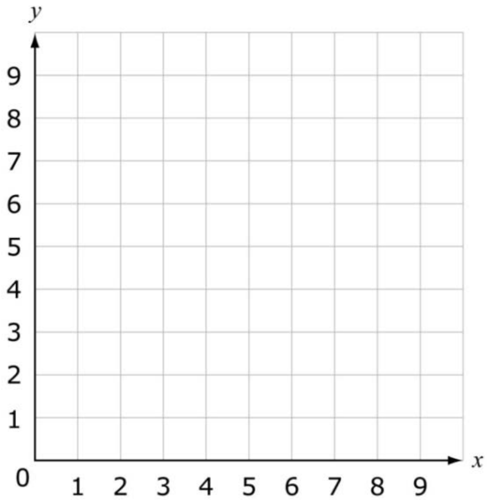
Examples:

- On a map, the library is located at (-2, 2), the city hall building is located at (0,2), and the high school is located at (0,0). Represent the locations as points on a coordinate grid with a unit of 1 mile.
  - What is the distance from the library to the city hall building? The distance from the city hall building to the high school? How do you know?
  - What shape is formed by connecting the three locations? The city council is planning to place a city park in this area. How large is the area of the planned park?

Part A

On the coordinate grid, plot the following points in order and connect each plotted point to the previous one in the order shown to form a figure.

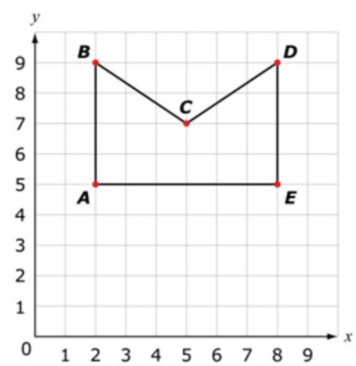
1. Point A (2, 5)
2. Point B (2, 9)
3. Point C (5, 7)
4. Point D (8, 9)
5. Point E (8, 5)
6. Point A (2, 5)



Part B

What is the area, in square units, of the enclosed figure?

Solution: ***Part A***



Part B 18 square units

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.G.A.3
    - Polygons in the Coordinate Plane
    - Walking the Block

MARS:

- Using Coordinates to Interpret and Represent Da

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- Plotter the Penguin

Nrich Mathematics:

- What are You Plotting?
- Coordinate Patterns
- Marbles in a Box
- Coordinate Tan
- Coordinate Challenge
- A Cartesian Puzzle

Common Misconceptions:

Common errors when plotting points in the coordinate plane include transposing the *x* and *y*-coordinates, mistaking a vertical or horizontal line on the plane by miscounting or struggling visually with the difference between the lines, and confusing the positive and negative parts of the perpendicular number lines when plotting points.

Domain: Geometry (G)

◆ **Cluster A:** *Solve real-world and mathematical problems involving area, surface area, and volume*.

Standard: 6.G.4

Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems. **(6.G.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.G.1

Explanations and Examples: 6.G.4

A net is a two-dimensional representation of a three-dimensional figure. Students represent three-dimensional figures whose nets are composed of rectangles and triangles. Students recognize that parallel lines on a net are congruent. Using the dimensions of the individual faces, students calculate the area of each rectangle and/or triangle and add these sums together to find the surface area of the figure.

Students construct models and nets of three dimensional figures, describing them by the number of edges, vertices, and faces. Solids include rectangular and triangular prisms. Students are expected to use the net to calculate the surface area.

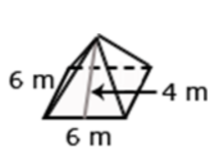
Students can create nets of 3D figures with specified dimensions using the Dynamic Paper Tool on NCTM's Illuminations.

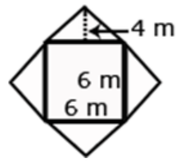
Students also describe the types of faces needed to create a three-dimensional figure. Students make and test conjectures by determining what is needed to create a specific three-dimensional figure.

Examples:

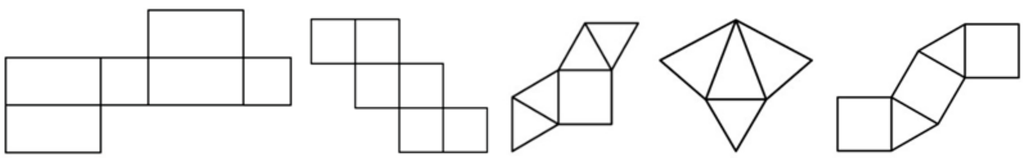
- Describe the shapes of the faces needed to construct a rectangular pyramid. Cut out the shapes and create a model. Did your faces work? Why or why not?

- Create the net for a given prism or pyramid, and then use the net to calculate the surface area.



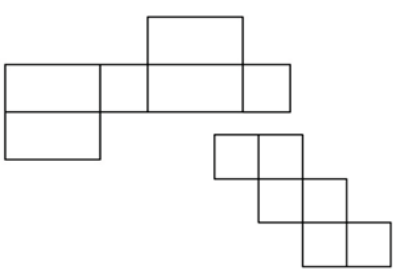
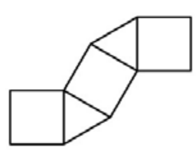
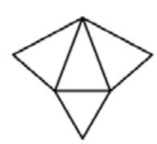
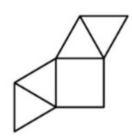


- Classify each net as representing a rectangular prism, a triangular prism, or a pyramid. To place an object in a region, click the object, move the pointer over the region, and click again to place the object in the region. To return all objects to their original positions, click the Reset button.



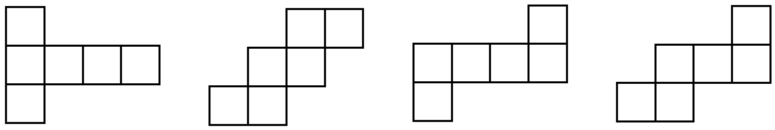
Nets Forming a Rectangular Prism	Nets Forming a Triangular Prism	Nets Forming a Pyramid

Solution:

Nets Forming a Rectangular Prism	Nets Forming a Triangular Prism	Nets Forming a Pyramid
		 

Instructional Strategies: 6.G.4

Understanding that there are multiple nets for the same object may be difficult for some to visualize. Provide concrete examples of nets for the object. Both the composition and decomposition of rectangular prisms should be explored. For example; the following are a few of the possible nets that will create a cube.



Resources/Tools:

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- Cube Nets
- Dynamic Paper

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.G.A.4
    - Nets for Pyramids and Prisms

MARS:

- Designing 3D Products: Candy Carton

Robert Kaplinksy's Lessons:

- How Much Does the Paint on the Space Shuttle Weigh?
- How Much Does the Aluminum Foil Prank Cost?

Nrich Mathematics:

- Cut Nets
- Instant Insanity

Common Misconceptions: See 6.G.1

Domain: Statistics and Probability (SP)

● **Cluster A:** *Develop concepts of statistical measures of center and variability and an informal understanding of outlier.*

Standard: 6.SP.1

Recognize and generate a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.* **(6.SP.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: **6.SP.1 through 6.SP.3**

This cluster is connected to:

- Developing an understanding of statistical thinking.
- Measures of center and measures of variability are used to draw informal comparative inferences about two populations in Grade 7 - Statistics and Probability.

Explanations and Examples: 6.SP.1

Students differentiate between statistical questions and those that are not. A statistical question is one that collects information that addresses differences in a population. The question is framed so that the responses will allow for the differences. For example, the question, "How tall am I?" is not a statistical question because there is only one response; however, the question, "How tall are the students in my class?" is a statistical question since the responses would allow for differences.

Statistics are numerical data relating to an aggregate of individuals; statistics is also the name for the science of collecting, analyzing and interpreting such data. A statistical question anticipates an answer that varies from one individual to the next and is written to account for the variability in the data. Data are the numbers produced in response to a statistical question. Data are frequently collected from surveys or other sources (i.e. documents).

Questions can result in a narrow or wide range of numerical values. For example, asking classmates "How old are the students in my class in years?" will result in less variability than asking "How old are the students in my class in months?"

Students might want to know about the fitness of the students at their school. Specifically, they want to know about the exercise habits of the students. So rather than asking "Do you exercise?" they should ask about the amount of exercise the students at their school get per week. A statistical question for this study could be: "How many hours per week on average do students at Jefferson Middle School exercise?"

To collect this information, students might design a survey question that anticipates variability by providing a variety of possible anticipated responses that have numerical answers, such as: 3 hours per week, 4 hours per week, and so on. Be sure that students ask questions that have specific numerical answers.

Instructional Strategies: 6.SP.1 though 6.SP.3

Grade 6 is the introduction to the formal study of statistics for students. Students need multiple opportunities to look at data to determine and word statistical questions. Data should be analyzed from many sources, such as organized lists, box-plots, bar graphs and stem-and-leaf plots. This will help students begin to understand that responses to a statistical question will vary, and that this variability is described in terms of spread and overall shape. At the same time, students should begin to relate their informal knowledge of mean, mode and median to understand that data can also be described by single numbers. The single value for each of the measures of center (mean, median or mode) and measures of spread (range and/or interquartile range) is used to summarize the data. Given measures of center for a set of data, students should use the value to describe the data in words.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.SP.A.1
  - Buttons: Statistical Questions
  - Identifying Statistical Questions
  - Statistical Questions

Common Misconceptions: 6.SP.1 through 6.SP.3

Students may believe all graphical displays are symmetrical. Exposing students to graphs of various shapes will show this to be false.

Domain: Statistics and Probability (SP)

● **Cluster A:** *Develop concepts of statistical measures of center and variability and an informal understanding of outlier.*

Standard: 6.SP.2

Recognize and generate a statistical question with a distribution which can be described by its center (mean, median and/or mode), spread (range and/or interquartile range), and overall shape (cluster, peak, gap, symmetry, skew (data) and/or outlier). **(6.SP.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

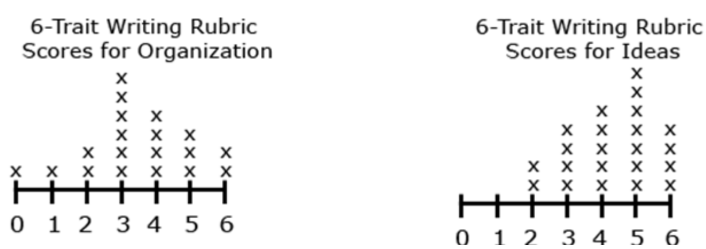
Connections: See 6.SP.1.

Explanations and Examples: 6.SP.2

The distribution is the arrangement of the values of a data set. Distribution can be described using **center** (**median**, **mode**, or **mean**), and **spread**. Data collected can be represented on graphs, which will show the shape of the distribution of the data. Students examine the distribution of a data set and discuss the center, spread and overall shape with dot plots, histograms and box plots.

The two dot plots show the 6-trait writing scores for a group of students on two different traits, organization and ideas. The center, spread and overall shape can be used to compare the data sets. Students consider the context in which the data were collected and identify clusters, peaks, gaps, and symmetry.

Showing the two graphs vertically rather than side by side helps students make comparisons. For example, students would be able to see from the display of the two graphs that the ideas scores are generally higher than the organization scores. One observation students might make is that the scores for organization are clustered around a score of 3 whereas the scores for ideas are clustered around a score of 5.



Instructional Strategies: 6.SP.2

Students should begin to relate their informal knowledge of **mean**, **mode** and **median** to understand that data can also be described by single numbers. The single value for each of the measures of center (mean, median or mode) and measures of spread (**range** and/or **interquartile range**) is used to summarize the data. Given measures of center for a set of data, students should use the value to describe the data in words.

The important purpose of the number is not the value itself, but the interpretation it provides for the variation of the data. Interpreting different measures of center for the same data develops the understanding of how each measure sheds a different light on the data. The use of a similarity and difference matrix to compare mean, median, mode and range may facilitate understanding the distinctions of purpose between and among the measures of center and spread. Use newspaper and magazine graphs for analysis of spread, shape and variation of data.

Include activities that require students to match graphs and explanations, or measures of center and explanations prior to interpreting graphs based upon the computation measures of center or spread.

The determination of the measures of center and the process for developing graphical representation is the focus of the cluster "Summarize and describe distributions" in the Statistics and Probability domain for Grade 6. Classroom instruction should integrate the two clusters.

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.SP.A.2
  - Puppy Weights
  - Electoral College
  - Average Number of Siblings
  - Is It Center or Is it Variability?
  - Describing Distributions

MARS:

- Representing Data with Grouped Frequency Graphs and Box Plots

Open Middle:

- Statistics and Probability Tasks

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- Histogram Tool

Nrich Mathematics:

- Statistics Tasks

Common Misconceptions: 6.SP.2

See also 6.SP.1

Students may not understand that the value of a measure of center describes the data, rather than a value used to interpret and describe the data.

Students will sometimes confuse how to describe skew. When the graph displays a large 'hump' (larger frequency of data) on the left hand side of the graph and the 'tail' (smaller frequency of data) is to the right, the skew is to the right. Students will try to describe the skew by looking at the larger frequency rather than the location of the tail.

Students will sometimes miscalculate the mean by forgetting to divide. This is usually because of a lack of understanding behind the meaning of this measure (that it is an evening out of the data) so they are unable to check the reasonability of their answer.

When finding the upper and lower quartiles in box plots, students will use median of the data set. They should use the data point to the left of the median to find the lower quartile and the data point to the right of the median to find the upper quartile.

Domain: Statistics and Probability (SP)

● **Cluster A:** *Develop concepts of statistical measures of center and variability and an informal understanding of outlier.*

Standard: 6.SP.3

Recognize that a measure of center (mean, median and/or mode) for a numerical data set summarizes all of its values with a single number, while a measure of variation (range and/or interquartile range) describes how its values vary with a single number. **(6.SP.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 6.SP.1

Explanations and Examples: 6.SP.3

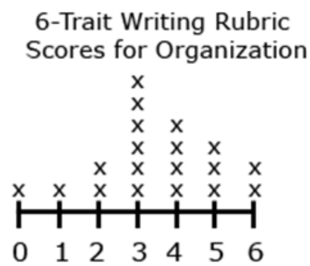
Data sets contain many numerical values that can be summarized by one number such as a measure of center. The measure of center gives a numerical value to represent the center of the data (i.e. midpoint of an ordered list or the balancing point). Another characteristic of a data set is the variability (or spread) of the values. Measures of variation are used to describe this characteristic.

When using measures of center (**mean**, **median**, and **mode**) and range, students are describing a data set in a single number. The range provides a single number that describes how the values vary across the data set. The range can also be expressed by stating the minimum and maximum values.

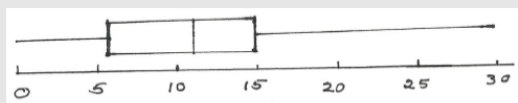
Quartiles, the medians of the lower and upper halves of the ordered data values, mark off the middle 50% of the data values, and so, provide information on the spread of the data. The distance between the first and third quartiles, the interquartile range, is a single number summary that serves as a very useful measure of variability.

Example:

- Consider the data shown in the dot plot of the six trait scores for organization for a group of students.
    - How many students are represented in the data set?
    - What are the mean, median, and mode of the data set? What do these values mean? How do they compare?
    - What is the range of the data? What does this value mean?



Box plot



For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 30}, the median is 11 (from the average of the two middle values 10 and 12), the interquartile range is $15 - 6 = 9$*, and the extreme values are 1 and 30.*

Instructional Strategies: See 6.SP.1

Resources/Tools:
See engageNY Modules 6 & 7

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.SP.A.3
    - Is It Center or Is It Variability?

MARS:

- Representing Variability with Mean, Median and Mode

Open Middle:

- Tasks for Statistics and Probability

Common Misconceptions: See 6.SP.1

Domain: Statistics and Probability (SP)

● **Cluster B:** *Summarize and describe distributions.*

Standard: 6.SP.4

Display numerical data on dot plots, histograms, stem-and-leaf plots, and box plots. **(6.SP.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: **6.SP.4-5**

This cluster is connected to:

- Developing understanding of statistical thinking.
- Measures of center and measures of variability are used to draw informal comparative inferences about two populations in Grade 7 - Statistics and Probability.

Explanations and Examples: 6.SP.4

Dot plots, histograms, box plots, and stem and leaf plots are four graphs to be used with this particular standard.

- A dot plot is a graph that uses a point (dot) for each piece of data. The plot can be used with data sets that include fractions and decimals. Dot plots are simple plots on a number line where each dot represents a piece of data in the data set. Dot plots are suitable for small to moderate size data sets and are useful for highlighting the distribution of the data including clusters, gaps, and outliers.
- A histogram shows the distribution of continuous data using intervals on the number line. The height of each bar represents the number of data values in that interval.
- A box plot shows the distribution of values in a data set by dividing the set into quartiles. The box plot is constructed from the five-number summary (minimum, lower quartile, median, upper quartile, and maximum). These values give a summary of the shape of a distribution. Students understand that the size of the box or whiskers represent the variability or spread. Students can readily compare two sets of data if they are displayed with side by side box plots on the same scale. Box plots display the degree of spread of the data and the skewness of the data. Box plots can be plotted horizontally or vertically on a number line.
- Stem and leaf plots are a method for showing the frequency with which certain classes of values occur. Each data value is split into a "stem" (the first digit or digits) and a "leaf" (usually the last digit). The "stem" values are listed down, and the "leaf" values go right (or left) from the stem values. The "stem" is used to group the values and each "leaf" shows the individual values within each group.

In most real data sets, there is a large amount of data and many numbers will be unique. A graph (such as a dot plot) that shows how many ones, how many twos, etc. would not be meaningful; however, a histogram can be used. Students organize the data into convenient ranges and use these intervals to generate a frequency table and histogram.

Note that changing the size of the range changes the appearance of the graph and the conclusions you may draw from it.

In order to display numerical data in dot plots, histograms or box plots, students need to make decisions and perform calculations. Students are expected to display data graphically in a format appropriate for that data set as well as reading data from graphs generated by others students or contained in reference materials. Students can use applets to create data displays.

- Box Plot Tool
- Histogram Tool

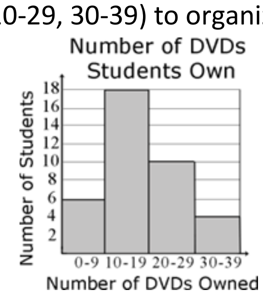
Examples:

- Nineteen students completed a writing sample that was scored using the six traits rubric. The scores for the trait of organization were 0, 1, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6. Create a data display. What are some observations that can be made from the data display?

- Grade 6 students were collecting data for a math class project. They decided they would survey the other two grade 6 classes to determine how many DVDs each student owns. A total of 48 students were surveyed. The data are shown in the table below in no specific order. Create a data display. What are some observations that can be made from the data display?

11	21	5	12	10	31	19	13	23	33
10	11	25	14	34	15	14	29	8	5
22	26	23	12	27	4	25	15	7	
2	19	12	39	17	16	15	28	16	

- A histogram using 4 ranges (0-9, 10-19, 20-29, 30-39) to organize the data is displayed below.



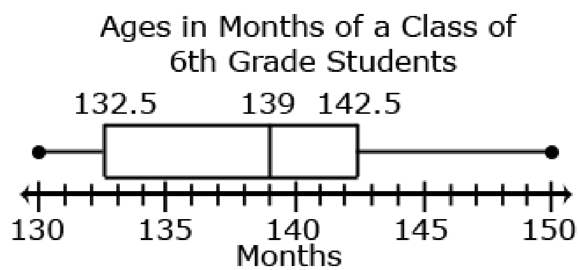
- A stem and leaf plot using the values: 12, 13, 21, 27, 33, 34, 35, 37, 40, 40, 41 is displayed below.

stem	leaf
1	2 3
2	1 7
3	3 4 5 7
4	0 0 1

- Ms. Wheeler asked each student in her class to write their age in months on a sticky note. The 28 students in the class brought their sticky note to the front of the room and posted them in order on the white board. The data set is listed below in order from least to greatest.
  Create a data display. What are some observations that can be made from the data display?

130	130	131	131	132	132	132	133	134	136
137	137	138	139	139	139	140	141	142	142
142	143	143	144	145	147	149	150		

Five number summary
Minimum – 130 months
Quartile 1 (Q1) – (132 + 133) ÷ 2 = 132.5 months
Median (Q2) – 139 months
Quartile 3 (Q3) – (142 + 143) ÷ 2 = 142.5 months
Maximum – 150 months

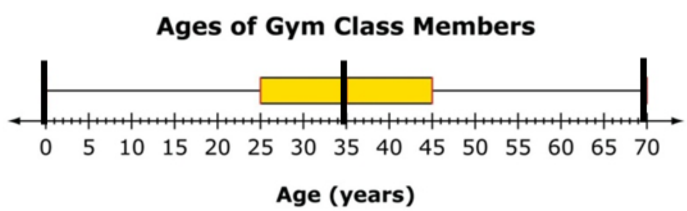


This box plot shows that:

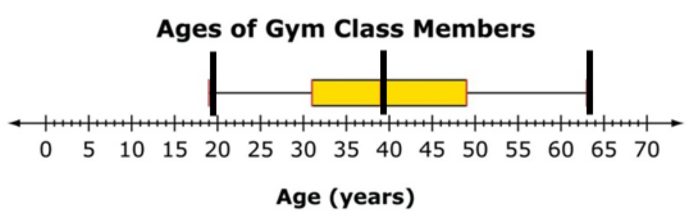
- $\frac{1}{4}$ of the students in the class are from 130 to 132.5 months old.
- $\frac{1}{4}$ of the students in the class are from 142.5 months to 150 months old.
- $\frac{1}{2}$ of the class are from 132.5 to 142.5 months old.
- The median class age is 139 months.

- The ages, in years, of the 28 members of a gym class are listed.
  19, 21, 22, 27, 29, 31, 31, 31, 33, 34, 37, 38, 39, 39,
  39, 41, 43, 45, 46, 47, 49, 49, 51, 51, 52, 54, 56, 63

Construct a box plot of the data in the list above. Click each **bold** line in the box plot and drag it to the correct position.



Solution:



Major Clusters Supporting Clusters Additional Clusters

Instructional Strategies: 6.SP.4

This cluster builds on the understandings developed in standards **6.SP.1 through 6.SP.3**. Students have analyzed data displayed in various ways to see how data can be described in terms of variability. Additionally, in Grades 3-5 students have created scaled picture and bar graphs, as well as line plots. Now students learn to organize data in appropriate representations such as box plots (box-and-whisker plots), dot plots, and stem-and-leaf plots. Students need to display the same data using different representations.

By comparing the different graphs of the same data, students develop understanding of the benefits of each type of representation.

Further interpretation of the variability comes from the range and center-of-measure numbers. Prior to learning the computation procedures for finding mean and median, students will benefit from concrete experiences.

To find the median visually and kinesthetically, students should reorder the data in ascending or descending order, then place a finger on each end of the data and continue to move toward the center by the same increments until the fingers touch. This number is the median.

The concept of mean (concept of fair shares or "*evening out*") can be demonstrated visually and kinesthetically by using stacks of linking cubes. The blocks are redistributed among the towers so that all towers have the same number of blocks. Students should not only determine the range and centers of measure, but also use these numbers to describe the variation of the data collected from the statistical question asked. The data should be described in terms of its shape, center, spread (range) and interquartile range. Providing activities that require students to sketch a representation based upon given measures of center and spread and a context will help create connections between the measures and real-life situations.

Continue to have students connect contextual situations to data to describe the data set in words prior to computation. Therefore, determining the measures of spread and measures of center mathematically need to follow the development of the conceptual understanding. Students should experience data which reveals both different and identical values for each of the measures. Students need opportunities to explore how changing a part of the data may change the measures of center and measure of spread.

Also, by discussing their findings, students will solidify understanding of the meanings of the measures of center and measures of variability, what each of the measures do and do not tell about a set of data, all leading to a better understanding of their usage.

Using graphing calculators to explore box plots (box-and-whisker plots) removes the time intensity from their creation and permits more time to be spent on the meaning. It is important to use the interquartile range in box plots when describing the variation of the data. Patterns in the graphical displays should be observed, as should any outliers in the data set. Students should identify the attributes of the data and know the appropriate use of the attributes when describing the data. Pairing contextual situations with data and its box-and-whisker plot is essential.

Resources/Tools:

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- Box Plots & Histograms
- Mean and Median

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.SP.B.4
    - Puppy Weights
    - Puzzle Times
    - Average Number of Siblings
    - Comparing Test Scores
    - Describing Distributions

Open Middle:

- Statistics and Probability tasks

Common Misconceptions:

Students often use words to help them recall how to determine the measures of center. However, student's lack of understanding of what the measures of center actually represent tends to confuse them. Median is the number in the middle, but that middle number can only be determined after the data entries are arranged in ascending or descending order. Mode is remembered as the "most," and often students think this means the largest value, not the "most frequent" entry in the set.

Academic vocabulary is important in mathematics and equally important is the development of conceptual understanding. Usually the mean, mode, or median have different values, but sometimes those values are the same. Students need to understand these terms and know more than just their definition or the algorithm for finding these values.

Domain: Statistics and Probability (SP)

● **Cluster B:** *Summarize and describe distributions.*

Standard: 6.SP.5

Summarize numerical data sets in relation to their context, such as by:

6.SP.5a. Reporting the number of observations. **(6.SP.5a)**

6.SP.5b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement. **(6.SP.5b)**

6.SP.5c. Giving quantitative measures of center (mean, median and/or mode) and variability (range and/or interquartile range), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered. **(6.SP.5c)**

6.SP.5d. Relating the choice of measures of center and variability to the distribution of the data. **(6.SP.5d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 6.SP.4

Explanations and Examples: 6.SP.5

Students record the number of observations. Using histograms, students determine the number of values between specified intervals. Given a box plot and the total number of data values, students identify the number of data points that are represented by the box. Reporting of the number of observations must consider the attribute of the data sets, including units (when applicable). Consideration may need to be given to how the data was collected (i.e. random sampling)

Given a set of data values, students summarize the measure of center with the median or mean. The median is the value in the middle of an ordered list of data. This value means that 50% of the data is greater than or equal to it and that 50% of the data is less than or equal to it.

The mean is the arithmetic average or balance point of a distribution. The mean is the sum of the values in a data set divided by how many values there are in the data set. The mean represents the value if all pieces of the data set had the same value. As a balancing point, the mean is the value where the data values above and the data values below have the same value.

Measures of variation can be described using the interquartile range. The interquartile range describes the variability between the middle 50% of a data set. It is found by subtracting the lower quartile from the upper quartile. It represents the length of the box in a box plot and is not affected by outliers. Students understand how the measures of center and measures of variability are represented by the graphical display.

Students describe the context of the data, using the shape of the data and are able to use this information to determine an appropriate measure of center and measure of variability.

Students summarize numerical data by providing background information about the attribute being measured, methods and unit of measurement, the context of data collection activities, the number of observations, and summary statistics. Summary statistics include quantitative measures of center, spread, and variability including extreme values (minimum and maximum), mean, median, mode, range, quartiles, and interquartile ranges.

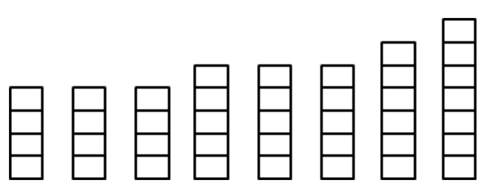
The measure of center that a student chooses to describe a data set will depend upon the shape of the data distribution and context of data collection. The mode is the value in the data set that occurs most frequently. The mode is the least frequently used as a measure of center because data sets may not have a mode, may have more than one mode, or the mode may not be descriptive of the data set. The mean is a very common measure of center computed by adding all the numbers in the set and dividing by the number of values. The mean can be affected greatly by a few data points that are very low or very high. In this case, the median or middle value of the data set might be more descriptive. In data sets that are symmetrically distributed, the mean and median will be very close to the same. In data sets that are skewed, the mean and median will be different, with the median frequently providing a better overall description of the data set.

Understanding the Mean

The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed **equally or "evened out"**, and also in the sense that it is a balance point. Students develop understanding of what the mean represents by redistributing data sets to be level or fair. The leveling process can be connected to and used to develop understanding of the computation of the mean.

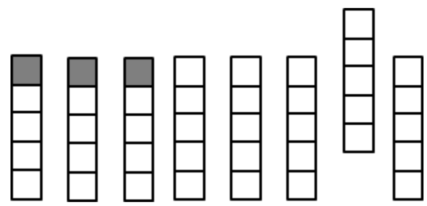
For example, students could generate a data set by measuring the number of jumping jacks they can perform in 5 seconds, the length of their feet to the nearest inch, or the number of letters in their names.

It is best if the data generated for this activity are 5 to 10 data points which are whole numbers between 1 and 10 that are easy to model with counters or stacking cubes. Students generate a data set by drawing eight student names at random from the Popsicle stick cup. The number of letters in each of the names is used to create the data set. If the names drawn were Carol, Mike, Maria, Luis, Monique, Sierra, John, and Karen there would be 3 names with 4 letters each, 3 names with 5 letters each, 1 name with 6 letters and 1 name with 7 letters. This data set could be represented with stacking cubes.



Students can model the mean by "leveling" the stacks or distributing the blocks so the stacks are "fair". Students are seeking to answer the question, "If all of the students had the same number of letters in their name, how many letters would each person have?"

One block from the stack of six and two blocks from the stack of 7 can be moved down to the stacks of 4 and then all the stacks have five blocks. If all students had the same number of letters in their name, they would have five letters. The mean number of letters in a name in this data set is 5.



If it was not possible to make the stacks exactly even, students could begin to consider what part of the extra blocks each stack would have.

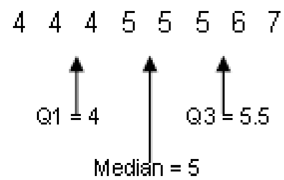
Understanding Medians and Quartiles

Students can also summarize and describe the center and variability in data sets using the median and a five number summary consisting of the minimum, quartiles, and maximum as seen in the box plot example in 6.SP.4. The median is the middle number of the data set with half the number below the median and half the numbers above the median. The quartiles partition the data set into four parts by dividing each of the halves of the data set into half again. Quartile 1 (Q1 or the lower quartile) is the middle value of the lower half of the data set and quartile 3 (Q3 or the upper quartile) is the middle value of the upper half of the data set. The median can also be referred to as quartile 2 (Q2). The range of the data is the difference between the minimum and maximum values. The interquartile range of the data is the difference between the lower and upper quartiles (Q3 – Q1). The interquartile range is a measure of the dispersion or spread of the data set: a small value indicates values that are clustered near the median whereas a larger value indicates values that are more distributed.

Consider the first data set again. Recall that the names drawn were Carol, Mike, Maria, Luis, Monique, Sierra, John, and Karen. The data set can be represented in a numerical list. To find the median and quartile, the values are placed in order from least to greatest.

5 4 5 4 7 6 4 5 ⟶ 4 4 4 5 5 5 6 7

The middle value in the ordered data set is the median. If there is an even number of values, the median is the mean of the middle two values. In this case, the median would be 5 because 5 is the average of the 4th and 5th values which are both 5. Students find quartile 1 (Q1) by examining the lower half of the data. Again there are 4 values which is an even number of values. Q1 would be the average of the 2nd and 3rd value in the data set or 4. Students find quartile 3 (Q3) by examining the upper half of the data. Q3 would be the average of the 6th and 7th value in the data set or 5.5. The mean of the data set was 5 and the median is also 5, showing that the values are probably clustered close to the mean. The interquartile range is 1.5 (5.5 – 4). The interquartile range is small, showing little variability in the data.



Instructional Strategies: See 6.SP.4

Resources/Tools:

Illustrative Mathematics Grade 6 tasks: **Scroll to the appropriate section to find named tasks.**

- 6.SP.B.5.c
  - Puzzle Times
  - Average Number of Siblings
  - Comparing Test Scores
  - Math Homework Problems
- 6.SP.B.5.d
  - Electoral College
  - Mean or Median?

Open Middle:

- Tasks for Statistics and Probability

Common Misconceptions: See 6.SP.4

APPENDIX: TABLE 1. Common Addition and Subtraction Situations

Shading taken from OA progression

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
	Total Unknown	**Addend Unknown**	**Both Addends Unknown[1]**
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[1]These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

[2]Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[3]For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

TABLE 2. Common Multiplication and Division Situations

Grade level identification of introduction of problem situations taken from OA progression

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	a × b = ?	a × ? = p, and p ÷ a = ?	? × b = p, and p ÷ b = ?

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change

TABLE 3. The Properties of Operations

the subject of the comparing sentence ("A red hat costs n times as much as the blue hat" results in the same comparison as "A blue hat is 1/n times as much as the red hat" but has a different subject.)

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a$ *and* $0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times 5)\times 2=32\times(5\times 2)$
Commutative	$a\times b=b\times a$	$10\times 38=38\times 10$
Multiplicative Identity	$a\times 1=a$ *and* $1\times a=a$	$387\times 1=387$
Multiplicative Inverse	*For every real number a,* $a\neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times 50+7\times 2$

(Variables a, b, and c represent real numbers.)
Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 4. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children

Levels	8 + 6 = 14	14 – 8 = 6
Level 1: Count all	Count All a: 1 2 3 4 5 6 7 8 b: 1 2 3 4 5 6 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b c
Level 2: Count on	Count On 8 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten 10 + 4 10 + 4	14 – 8: I make a ten for 8 + ? = 14 8 + 2 + 4 6 8 + 6 = 14
Doubles ± n	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) collection of items and understands that the last word tells "how many". Beyond on small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words.

Level 3 - At this level the student begins using known facts to solve for unknown facts. For example, the student uses "make a ten" where one addend breaks apart to make 10 with another addend OR a doubles plus/minus one strategy. Students begin to implicitly use the properties of operations.

Table 7: Cognitive Rigor Matrix/Depth of Knowledge (DOK)

The Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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