

Progressions for the Common Core State Standards for Mathematics

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Modeling, K–12

Overview

Mathematical models describe situations in the world, to the surprise of many. Albert Einstein wondered, “How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?”¹ This points to the basic reason to model with mathematics and statistics: to understand reality. Reality might be described by a law of nature governing the motion of an object dropped from a height above the ground^{A-CED.2} in terms of:

- the height above the ground of a person riding a Ferris wheel;^{F-TF.5}
- how people’s heights vary;^{S-ID.1}
- a risk factor for a disease;^{S-ID.5}
- the effectiveness of a medical treatment;^{S-ID.5}
- the amount of money in a savings account to which periodic additions are made;^{A-SSE.4}
- the unemployment rate.^{N-Q.1}

On a more sophisticated level, modeling the spread of an epidemic, assessing the security of a computer password, understanding cyclic populations of predator and prey in an ecosystem, finding an orbit for a communications satellite that keeps it always over the same spot, estimating how large an area of solar panels would be enough to power a city of a given size, understanding how global positioning systems (GPSs) work, estimating how long it would take to get to the nearest star—all can be done using mathematical modeling. A survey of how mathematics has impacted recent breakthroughs can be found in *Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century*.²

¹January 27, 1921, address to the Prussian Academy of Science, Berlin.

²This report was published in 2012 by the National Academies Press and can be read online at http://www.nap.edu/catalog.php?record_id=13373.

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

S-ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

S-ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

S-ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Mathematical modeling is fundamental to how mathematics is used in medicine, engineering, ecology, weather forecasting, oil exploration, finance and economics, business and marketing, climate modeling, designing search engines, understanding social networks, public key cryptography and cybersecurity, the space program, astronomy and cosmology, biology and genetics, criminology, using genetics to reconstruct how early humans spread over the planet, in testing and designing new drugs, in compressing images (JPEG) and music (MP3), in creating the algorithms that cell phones use to communicate, to optimize air traffic control and schedule flights, to design cars and wind turbines, to recommend which books (Amazon), music (Pandora) and movies (Netflix) an individual might like based on other things they rated highly. The range of careers for which mathematical and statistical modeling are good preparation has expanded substantially in recent years, and the list continues to grow.

Mathematical and statistical models of real world situations range in complexity from objects or drawings that represent addition and subtraction situations^{K.OA.2} to systems of equations that describe behaviors of natural phenomena such as fluid flow or the paths of ballistic missiles. Sometimes models give rather complete information about the situation. For example, writing total cost as the product of the unit price and the number bought is often a complete and accurate model of monetary costs. Some models do not give exact and complete information but approximations that may result from the features of the situation that are reasonably available or of most interest.^{MP.4} In the business world, the per item price when purchasing a large number of the same item is lower than for the price for a single item. This is important in modeling some situations but may be neglected in others. As another example, consider the linear function describing the cost of purchasing an automobile and gasoline for a number of years

$$C(t) = p + at,$$

where p is the purchase price, t is the number of years, and a is a constant based on assumptions of the cost of gasoline (per gallon), the number of miles driven per year and the fuel efficiency in miles per gallon.^{F-BF.1} All of the quantities going into the constant a are estimates and likely will not be constant over time, but a more complex model of gasoline costs and expected driving habits requires information not available and perhaps unnecessary for decision-making. Further, there are costs not included—insurance and maintenance, for example—and the effect of a gasoline-powered automobile on the environment is not considered. However, the simple model may suffice to decide, say, between the purchase of a hybrid version and a gasoline version of an automobile, where the basic differences are in purchase price (hybrids may cost more) and fuel efficiency. Following the advice attributed to Einstein that, “Every-

What is a model?

The word “model” can be used as a noun, verb, or adjective. As an adjective, “model” often signifies an ideal, as in “model student.” In this progression, “model” will be a noun or a verb.

In elementary mathematics, a model might be a representation such as a math drawing or a situation equation (operations and algebraic thinking), line plot, picture graph, or bar graph (measurement), or building made of blocks (geometry). In Grades 6–7, a model could be a table or plotted line (ratio and proportional reasoning) or box plot, scatter plot, or histogram (statistics and probability). In Grade 8, students begin to use functions to model relationships between quantities.

In high school, modeling becomes more complex, building on what students have learned in K–8. Representations such as tables or scatter plots are often intermediate steps rather than the models themselves.

The same representations and concrete objects used as models of real life situations are used to understand mathematical or statistical concepts. In elementary grades, students use rows of dots or tape diagrams to represent addition and subtraction of numbers as well as to model quantities in real life situations. Later they use tape diagrams, arrays, and area diagrams to represent multiplication and division of numbers and to model quantities. In Grade 6 geometry, nets may represent a three-dimensional mathematical object (e.g., a prism) as well as a design for a real world object (e.g., a gingerbread house). In Grade 8, students use physical objects (e.g., paper triangles, transparencies, or geometry software to understand congruence and similarity. In Grade 6–8 statistics, simulations help students to understand what can happen during statistical sampling.

The use of representations and physical objects to understand mathematics is sometimes referred to as “modeling mathematics,” and the associated representations and objects are sometimes called “models.” For example, an area diagram is often called an “area model” when it is used to describe 2×3 as well as when it is used to describe a rectangular garden that is 2 feet by 3 feet. Students need not make distinctions such as “area diagram” and “area model.” However, readers of the Progressions should be aware that “modeling” as used in the Standards is primarily about using mathematics to describe the real world.

K.OA.2 Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

MP.4 Mathematically proficient students . . . are comfortable making assumptions and approximations . . . realizing that these may need revision later.

F-BF.1 Write a function that describes a relationship between two quantities.

thing should be made as simple as possible, but not simpler;³ we can get good evidence to support the choice of the simple model $C(t) = p + at$, so in this situation this is “as simple as possible,” but dropping the purchase price p , or the term at would delete critical information for our decision, based on cost differences. That would delete necessary information, moving us to Einstein’s “simpler [than possible].”

Models mimic features of reality. These features are often selected for particular uses. For example, a road map is a model. So is a geological map. Features that are important on road maps, e.g., major highways, may not be important on a geological map. The features of the real world situation mimicked by a mathematical model fall into three categories:⁴

- Things whose effects are neglected.
- Things that affect the model but whose behavior the model is not designed to study—inputs or independent variables.
- Things that the model is designed to study—outputs or dependent variables. •

- Statistical modeling also involves relationships among variables, but the relationship may be construed as association (e.g., correlation) rather than dependency.

These features of a mathematical model are helpful to keep in mind. For example, in the cost function C above, the effect on environment, insurance costs, and maintenance costs are neglected. Inputs are cost of gasoline, miles driven per year, and fuel efficiency rate. The output, or dependent variable, is the cost.

Modeling in K–12

Modeling is critically important, but is not easy. Some idealized, simple modeling problems are needed for learning throughout K–12, but real problems are easily available and solvable (perhaps with the assistance of technology). Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to represent purely mathematical phenomena as well as to model physical phenomena. Situations which are not modeled by simple equations can often be understood by simulation on a calculator, desktop, or laptop, a process which many students will find especially engaging because of its exploratory and open-ended nature. These tools allow for modeling complex real world situations, and most real world situations are complex.

While there is certainly no limit to the sophistication of a model or of the mathematics used in a model, the essence of modeling is

³Probably a paraphrase of “It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience” from “On the Method of Theoretical Physics.”

⁴Bender, 1978, *An Introduction to Mathematical Modeling*, John Wiley and Sons.

often to use humble mathematics in rather sophisticated ways. For example, percentages are often crucial in modeling situations. “Distance equals rate times time”^{6.EE.9} is a type of powerful idea that is introduced in Grade 6 that nevertheless forms the basis for many useful models throughout high school and beyond. Or as another example, when high school students make an order of magnitude estimate, they may learn a great deal by using only simple multiplication and division. Likewise, statistical modeling in high school might often involve only measures of center and variability, rather than relying on a host of sophisticated statistical techniques. “Back of the envelope” modeling is one of the discipline’s most powerful forms.

Many situations in the real world involve rate of change, with models that involve a differential equation. Although differential equations are not in the Standards, the interpretation of rates of change^{S-ID.7} and the study of functions with base rules of growth^{F-LE.1} prepares the way for the study of more sophisticated models in college. Likewise, using probability in modeling greatly extends the scope of real world situations which can be modeled.

News media accounts of topics of current interest often illustrate why modeling and understanding the models of others is important, mostly for informed citizenship. For example, probabilities often are stated in terms of odds in media accounts. Thus, to connect such accounts to school mathematics, students need to know the relationship between the two. Learning to model and understand models is enhanced by seeing the same mathematics or statistics model situations in different contexts. • Media accounts provide those varied contexts in circumstances that require critical thinking. Analyzing these accounts provides opportunities for students to maintain and deepen their understanding of modeling in high school and after graduation.⁵

The modeling process

In the Standards, modeling means using mathematics or statistics to describe (i.e., model) a real world situation and deduce additional information about the situation by mathematical or statistical computation and analysis. For example, if the annual rate of inflation is assumed to be 3% and your current salary is \$38,000 per year, what is an equivalent salary t years in the future? What salary is equivalent in 10 years? The model is familiar to many:

$$S(t) = 38,000(1.03^t).$$

This aspect of modeling produces information about the real world situation via the mathematical model, i.e. the real world is understood through the mathematics.

⁵For further examples, see Dingman & Madison, 2010, “Quantitative Reasoning in the Contemporary World” [two-part article], *Numeracy*, <http://scholarcommons.usf.edu/numeracy/vol3/iss2/>.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

F-LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

- For example, right triangles are a frequent model for situations that students may initially see as different mathematically, e.g., finding the length of the shadow cast by an upright pole and finding the height of a tree or building. A line fitted to a scatter plot is often used in statistics to model relationships between two measurement quantities. Risk factors are often derived from relative frequency within a single sample.

Complex models are often built hierarchically, out of simpler components which can then be artfully joined together to capture the behavior of the complex system. Certain simplifications have become standard based on historical use. For example, the consumer price index (CPI) and the cost of living index (COLI) are commonly cited measures that serve as agreed-upon proxies for important economic circumstances, substituting a single quantity for a more complicated collection of quantities that tend to move as a group. There is even an index of indexes, the index of leading economic indicators. The monthly payments required to amortize a home mortgage over 30 years are computed by summing a geometric series and manipulating the results.^{A-SSE.4} Numerous political and economic debates center on how one measures amounts of money, that is, what units are used. Measuring amounts of money in nominal dollars (dollars-of-the-day) over periods of several years is very different from measuring in constant dollars (the dollar of a particular year). Measuring in percent of gross domestic product (GDP) is also different. Understanding what these are and how to move from one unit to the others is critical in understanding many issues important to personal prosperity and responsible citizenship.

Probability and statistical models abound in news media reports. Complex and heretofore unusual graphics are made possible by technology and in recent years the diversity of graphical models in media accounts has increased enormously. Many of these models and the situations they describe are very important for making decisions about health issues or political circumstances. Political polls model elections themselves,^{S-IC.1} and skeptics decry their predictions because they are based on a small sample of all eligible voters. Lack of understanding leads to suspicion and distrust of democratic processes.

Modeling in high school

The modeling cycle

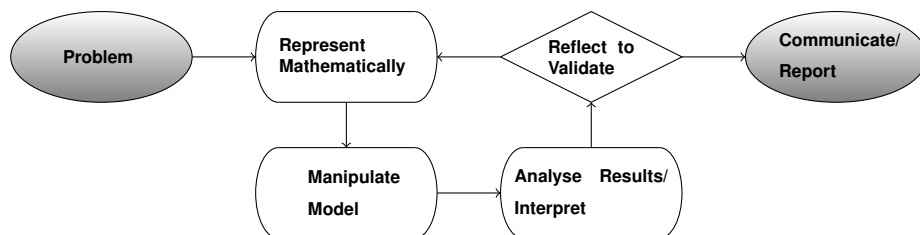
In high school, modeling involves a way of thought different from what students are taught when they learn much of the core K–8 mathematics. It provides experience in approaching problems that are not precisely formulated and for which there is not necessarily a single “correct” answer. Deciding what is left out of a model can be as important as deciding what is put in. Judgment, approximation, and critical thinking enter into the process. Modeling can have differing goals depending on the situation—sometimes the aim is quantitative prediction, for example in weather modeling, and sometimes the aim is to create a simple model that captures some qualitative aspect of the system with a goal of better understanding the system, for example modeling the cyclic nature of predator-prey populations.

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

S-IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

Why is modeling difficult? Modeling requires multiple mental activities and significant human skills of abstraction, analysis, and communication. First, a real world situation must be understood in terms familiar to the student. Critical variables must be identified and those that represent essential features are selected. Second, the interpreted situation must be represented—by diagrams, graphs, equations, or tables. Moving from the interpretation to the representation involves reasoning—algebraic, proportional, quantitative, geometric, or statistical. Symbolic manipulation and calculation^{A-SSE.3} may follow to produce expressions for the desired quantities. A critical step is now to interpret the quantitative information in terms of the original situation. The quantitative information must be analyzed or synthesized, that is, information is either combined to make some judgment or separated into pieces to do so. During this analysis or synthesis, assumptions are either made or assumptions are evaluated. At this point, the information obtained is evaluated in terms of the original situation. If the information is unreasonable or inadequate, then the model may need to be modified to re-start the whole process. If the information is reasonable and adequate, the results are communicated in terms reflecting the original real world context and the information sought by the student. Understanding the limitations of the model involves critical thinking.

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.



This figure is a variation of the figure in the introduction to high school modeling in the Standards.

Diagrams of modeling processes vary. For example, a diagram that focuses on reasoning processes has four components: Description, Manipulation, Translation or Prediction, and Verification.⁶ Partitioning the modeling process into reasoning components is helpful in identifying where reasoning is succeeding or failing. This is important in both assessing student work and guiding instruction. These diagrams of modeling processes are intended as guides for teachers and curriculum developers rather than as illustrations of steps to be memorized by students.

⁶See Lesh & Doerr, 2003, *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching*, Lawrence Erlbaum Associates, p. 17.

Units and modeling

Throughout the modeling process, units are critical for several reasons, including guiding the symbolic or numeric calculations.^{N-Q.1} Keeping track of units is very helpful in determining if the calculations are meaningful and lead to the desired results. Units are also critical in the analysis and synthesis and in making or evaluating assumptions, as well as determining reasonableness of answer. For example, if analysis of a cost equation for driving an automobile indicates that a typical driver in the U.S. will drive 5000 miles per year, one should check units to make sure that the gallons are U.S. gallons and the fuel efficiency is in miles per U.S. gallon. Most of the world measures gasoline in liters and distances in kilometers rather than miles. (According to the Federal Highway Administration, the average number of miles driven per year by a U.S. driver is over 13,000.) Units are almost always essential in communicating the results of a model since answers to real world problems are usually quantities, that is, numbers with units. Modeling prior to high school produces measures of attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling such as acceleration, percent of GDP, person-hours, and some measures where the units are not specified and have to be understood in the way the measure is defined.^{N-Q.2} For example, the S&P 500 stock index is a measure derived from the quotient of the value of 500 companies now and in 1940–42.

N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

N-Q.2 Define appropriate quantities for the purpose of descriptive modeling.

Modeling and the Standards for Mathematical Practice

One of the practice standards—*Model with mathematics* (MP.4)—focuses on modeling, and modeling draws on and develops all eight. This helps explain why modeling with mathematics and statistics is challenging. It is a capstone, the proof of the pudding. To embody it, students might complete a capstone experience in modeling.

Make sense of problems and persevere in solving them (MP.1) begins with the essence of problem solving by modeling: “Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.” Solving a real life problem in a non-mathematical context by mathematizing (i.e. modeling) requires knowing the meaning of the problem and finding a mathematical representation. As described later in this standard, younger students might use concrete objects or pictures as models to help conceptualize and solve such problems.

Reason abstractly and quantitatively (MP.2) includes two critical modeling activities. The first is “the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate,” and the second is that “Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved.” Decontextualizing and representing are fundamental to problem solving by modeling.

Construct viable arguments and critique the reasoning of others (MP.3) notes that mathematically proficient students “reason inductively about data, making plausible arguments that take into account the context from which the data arose”—such reasoning might include educating a model from the data. Further, “Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.” When the objects, drawings, diagrams, or actions are used as models of real life situations, discussing the validity of the models and the level of uncertainty in the results makes use of these skills.

Use appropriate tools strategically (MP.5) notes that “When making mathematical models, [mathematically proficient students] know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.” Simulation provides an important path to explore the consequences of a model, and to see what happens when parameters of the model are varied.

Attend to precision (MP.6). Here the most important consideration of modeling is to “express numerical answers with a degree of precision appropriate for the problem context” and in appropriate units. For example, if one is modeling the annual debt or surplus (there were no surpluses) in the U.S. federal budget over the decade 2001–2010, then common options for a unit are nominal dollars, constant dollars, or percent of GDP. The degree of precision appropriate for understanding the model is to the nearest billion dollars (or nearest tenth percent of GDP) or perhaps the nearest ten billion dollars (or nearest percent of GDP).^{N-Q.3} Beyond accuracy, modeling raises the issue of uncertainty—how likely are the quantities we want to model to be within a certain range. How much do features the model neglects affect accuracy and uncertainty?

Look for and make use of structure (MP.7). Here, looking closely at a real world situation to discern relationships between quantities is critical for mathematical modeling. Students look for patterns or structure in the situation, for example, seeing the side of a right triangle when a shadow is cast by an upright flagpole as part of a right triangle or seeing the rise and run of a ramp on a staircase.

Look for and express regularity in repeated reasoning (MP.8). Modeling activities often involve multistep calculations and the whole modeling cycle may need to be repeated. Here, mathematically proficient students “continually evaluate the reasonableness of their intermediate results” and “maintain oversight of the process” (in this case, the modeling process).

N-Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Modeling and reasonableness of answers

Continually evaluating reasonableness of intermediate results in problem solving is important in several of the standards for mathematical practice. Doing this often requires having reference values, sometimes called anchors or quantitative benchmarks, for compar-

ison. Joel Best, in his book *Stat-Spotting*,⁷ lists a few quantitative benchmarks necessary for understanding U.S. social statistics: the U.S. population, the annual birth and death rates, and the approximate fractions of the minority subpopulations. Without these reference values, an answer of 27 million 18-year-olds in the U.S. population may seem reasonable. Such benchmarks for other measures are helpful, providing quick ways to mentally check intermediate answers while solving multistep problems. For example, it is very helpful to know that a kilogram is approximately 2 pounds, a meter is a bit longer than a yard, and there are about 3 liters in a gallon. This kind of quantitative awareness can be developed with practice, and easily expanded with the immense amount of information readily available from the internet.

Statistics and probability

Specific modeling standards appear throughout the high school standards indicated by a star symbol (★). About one in four of the standards in Number and Quantity, Algebra, Functions, and Geometry have a star, but the entire conceptual category of Statistics and Probability has a star. In statistics, students use statistical and probability models—whose data and variables are often embodied in graphs, tables, and diagrams—to understand reality. Statistical problem solving is an investigative process designed to understand variability and uncertainty in real life situations. Students formulate a question (anticipating variability), collect data (acknowledging variability), analyze data (accounting for variability), and interpret results (allowing for variability).⁸ The final step is a report.

Much of the study of statistics and probability in Grades 6–8 concerns describing variability, building on experiences with categorical and measurement data in early grades (see the progressions for these domains). In high school the focus shifts to drawing inferences—that is, conclusions—from data in the face of statistical uncertainty. In this process, analyzing data may have two steps: representing data and fitting a function (often called the model) which is intended to capture a relationship of the variables. For example, bivariate quantitative data might be represented by a scatter plot and then the scatter plot is modeled as a linear, quadratic, or logarithmic function. A probability distribution might be represented as a bar graph and then the bar graph is modeled by an exponential function. See the high school Statistics and Probability Progression for examples.

Because the Statistics and Probability Progression for high school is also a modeling progression, the discussion here will only note

⁷2008, *Stat-Spotting: A Field Guide to Identifying Dubious Data*, University of California Press.

⁸See the American Statistical Association's 2007 *Guidelines for Assessment and Instruction in Statistics Education*, Alexandria, VA: American Statistical Association, 2007, pp. 11–15, <http://www.amstat.org/education/gaise>.

statistics and probability standards when they are related to modeling standards in one of the other conceptual categories.

Developing high school modeling

In early grades, students model addition and subtraction relationships among quantities such as 2 apples and 3 apples. Concrete objects, drawings, numerical equations, and diagrams serve as models, helping to explain arithmetic as well as to represent addition, subtraction, multiplication, and division situations described in the Operations and Algebraic Thinking Progression. Later, students use graphs and symbolic equations to represent relationships among quantities such as the price of n apples where p is the price per apple. In Grade 8, calculating and interpreting the concept of slope may, in various contexts, draw on interpreting subtraction as measuring change or as comparison, and division as equal partition or as comparison (see Tables 2 and 3 of the Operations and Algebraic Thinking Progression). Creation of exponential models builds on initial understanding of positive integer exponents as a representation of repeated multiplication, while identifying the base of the exponential expression from a table requires the unknown factor interpretation of division. Extension of an exponential model from a geometric sequence to a function defined on the real numbers builds on the understanding of rational and irrational numbers developed in Grades 6–8 (see the Number System Progression).

By the beginning of high school, variables and algebraic expressions are available for representing quantities in a context. Modeling in high school can proceed in two ways. First, problems can focus directly on the concepts being studied, i.e., situations such as the path of a projectile which are modeled by quadratic equations can be a part of the study of quadratic equations. This is the traditional path followed by having a section of word problems at the end of a lesson. A second, more realistic, way to develop modeling is to utilize situations that can become more complex as more mathematics and statistics are learned.⁹ It is unlikely that one situation can be used throughout high school modeling, but some situations can be increased in complexity (examples are given in this progression). Modeling with mathematics in high school begins with linear and exponential models and proceeds to representing more complex situations with quadratics and other polynomials, geometric and trigonometric models, logic models such as flow charts, diagrams with graphs and networks, composite functional models such as logistic ones, and combinations and systems of these. Modeling with statistics and probability (that is, as noted earlier, essentially

⁹For examples, see Schoen & Hirsch, "The Core-Plus Mathematics Project: Perspectives and Student Achievement," and Senk, "Effects of the UCSMP Secondary School Curriculum on Students' Achievement" in Senk & Thompson (Eds.), 2003, *Standards-Based School Mathematics Curricula: What Are They? What Do Students, Learn?*, Lawrence Erlbaum Associates.

all of statistics and probability) is detailed in the progression for that conceptual category.

Linear and exponential models

In high school, the most commonly occurring relationships are those modeled by linear and exponential functions. Examples abound. The number of miles traveled in t hours by an automobile at a speed of 30 miles per hour is $30t$ and the amount of money in an account earning 4% interest compounded annually after 3 years is $P(1.04)^3$ where P is the initial deposit. Students learn to identify the referents of symbols within expressions (MP.2), e.g., 30 is the speed (or, later, velocity), t is the time in hours, and to abstract distance traveled as the product of velocity and time.

In Grade 8, students learned that functions are relationships where one quantity (output or dependent variable) is determined by another (input or independent variable).^{8.F.1} In high school, they deepen their understanding of functions, learning that the set of inputs is the domain of the function and the set of outputs is the range.^{F-IF.1} For example, the car traveling 30 miles per hour travels a distance d in t hours is expressed as a function

$$d(t) = 30t.$$

Students learn that when a function arises in a real world context a reasonable domain for the function is often determined by that context.

Students learn that functions provide ways of comparing quantities and making decisions. For example, a more fuel-efficient automobile costs \$3000 more than a less fuel-efficient one, and \$500 per year will be saved on gasoline with the more efficient car. (This can be made more realistic by using data, say, from comparing a hybrid version to a gasoline version of an automotive model.) A graph of the net savings function

$$S(t) = 500t - 3000$$

(see margin) will have a vertical intercept at $S = -3000$ and a horizontal intercept at $t = 6$. Students learn that the horizontal intercept (the zero of the function) is the break-even point, that is, by year 6 the \$3000 extra cost has been recovered in savings on gasoline costs.^{F-LE.5}

As students learn more about comparing functions that have domains other than the nonnegative integers, this example can be increased in complexity.¹¹ The buyer has the option of paying the extra \$3000 and saving money on gasoline or placing the \$3000 in

¹¹Example from Madison, Boersma, Diefenderfer, & Dingman, 2009, *Case Studies for Quantitative Reasoning*, Pearson Custom Publishing.

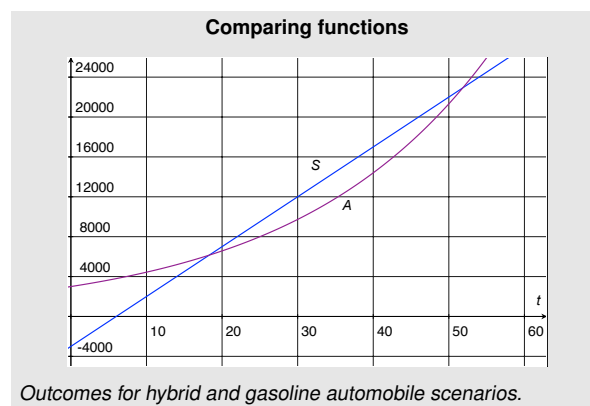
8.F.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.¹⁰

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x . The graph of f is the graph of the equation $y = f(x)$.

Comparing functions

Year t	$S(t)$	$A(t)$	Year t	$S(t)$	$A(t)$
0	-3000	3000	10	2000	4441
1	-2500	3120	11	2500	4618
2	-2000	3245	12	3000	4803
3	-1500	3375	13	3500	4995
4	-1000	3510	14	4000	5195
5	-500	3650	15	4500	5403
6	0	3796	16	5000	5619
7	500	3948	17	5500	5844
8	1000	4106	18	6000	6077
9	1500	4270	19	6500	6321

Outcomes for two scenarios. If the hybrid is purchased, its savings on gasoline costs plus the difference in price between hybrid and gasoline models is $S(t)$. If the gasoline model is purchased and the price difference is invested, the amount of the investment is $A(t)$.



F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

a savings account earning 4% per year compounded yearly. One option yields the net savings

$$S(t) = 500t - 3000$$

while the other yields amount

$$A(t) = 3000(1.04^t).$$

Students compare $S(t)$ and $A(t)$ by graphs or tables over some number of years, the domain of the functions.^{F-IF.9} The expected time the buyer will drive the car determines a reasonable domain. A table of values for A and S (shown in the margin) over years 1 to 20 is likely to be sufficient for comparing the functions,^{F-IF.6} or, later when non-integer domains are understood, the graphs of S and A over the interval $[0,20]$ will give considerable information (see the margin). The vertical intercepts of the two graphs and their two points of intersection are interpreted in the context of the problem. Analysis of the key features of the two graphs^{F-IF.4} provides opportunities for students to compare the behaviors of linear and exponential functions. Students observe the average rates of change of the two functions over various intervals^{F-IF.6} and see why the exponential function values will eventually overtake the linear function values and remain greater beyond some point. Students can now report on the information that will influence an economic decision by relating the behavior of the graphs to the comparative savings.

Students can again question the assumptions underlying the models of the two savings functions. What is the effect if the cost of gasoline changes? What is the effect if the number of miles driven changes? What will be the results of periodically (say, annually) placing the savings on gasoline costs in the savings account earning 4% per year compounded yearly? This latter option changes the linear model to a second exponential model, starts with a sum of a geometric series, which can be expressed either recursively or with an explicit formula,^{F-BF.2} and points to the advantages of rewriting the sum of exponential expressions as a single exponential expression.^{A-SSE.3c} This reinforces that algebraic re-writing of expressions is helpful, sometimes essential, to achieve comprehensible and usable models.

In the above example, students learn to question why the two scenarios have a \$6000 difference at year 0. Students might argue that the \$3000 is being invested two ways—one way is investing in the automobile and one way is placing in a savings account. The question then becomes: Which investment produces the most returns? That would make both functions be 0 at time 0. Is it more reasonable to note that the difference is \$3000 and not \$6000? In that case the graphs look like the ones here, and the table above is altered by reducing each entry for $A(t)$ by 3000.

Students learn that some initial representations and calculations can be done by hand,^{F-IF.7} say, the graph of $S(t) = 500t - 3000$ and

F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

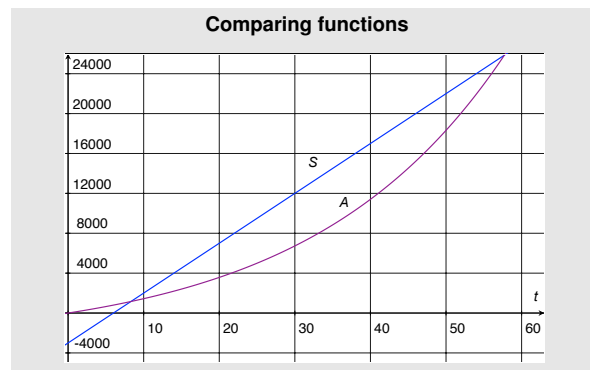
F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

A-SSE.3c Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

c Use the properties of exponents to transform expressions for exponential functions.



F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

its key features. With iterations of the modeling cycle, the model becomes more complicated. Specific outputs of the functions can be calculated by hand, but technology is essential to understand the overall situation.

Students learn to distinguish between scenarios like the one above where two (or more) equations or functions give different results based on different assumptions about the situation and scenarios where the two (or more) equations (possibly, inequalities) or functions express relationships among the quantities of interest under the same assumptions. The latter scenarios are modeled by a system of equations or inequalities. A system of equations imposes multiple conditions on a situation, one for each of the equations. Solutions to systems must satisfy each of the equations. For example, a system of two linear equations^{A-CED.3} will model the speed that you can row a boat with no current and the speed of the current provided you know the speed of the boat as you row with the current and the speed you can row against the current. Students learn how to describe situations by systems of two or three equations or inequalities and to solve the systems using graphs, substitution, or matrices. Students learn to detect if a system of equations is consistent, inconsistent, or independent.

Later, as students are challenged to develop more complex models, the processes of solving systems of equations are used to synthesize and develop new relationships from systems of equations that model a situation. Thus, students are challenged to use substitution to combine parametric equations and giving the spatial coordinates of a projectile as a function of time into a single relation modeling the path of the projectile, or to incorporate a constraint on the volume into a formula giving the cost of a cylindrical can as a function of the radius.

Counting, probability, odds and modeling

In Grades 7 and 8, students learned about probability and analysis of bivariate data. In high school, students learn the meanings of correlation and causation. Correlation, along with standard deviation, is interpreted in terms of a linear model of a data set. Students distinguish in models of real data the difference between correlation and causation.^{S-ID.7, S-ID.8, S-ID.9}

Students' intuitions, affected by media reports and the surrounding culture (cf. Nobel Laureate Daniel Kahnemann's *Thinking Fast and Slow*), sometimes conflict with their study of probability. Unusual events do occur and unconditional theoretical probabilities are based on what will happen over the long term and are not affected by the past—the probability of a head on a coin flip is $\frac{1}{2}$ even though each of the seven previous flips resulted in a head. Students learn how to reconcile accounts of probability from public and social media with their study of probability in school. For example, they learn the intriguing difference between conspiracy and coincidence.

A-CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

S-ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.

S-ID.9 Distinguish between correlation and causation.

Relating the study of probability to everyday language and feelings is important. Students learn about interpreting probabilities as “how surprised should we be?” Students learn to understand meanings of ordinary probabilistic words such as “unusual” by examples such as: “The really unusual day would be one where nothing unusual happens” and “280 times a day, a one-in-a-million shot is going to occur,” given that there were approximately 280 million people in the U.S. at the time. Coincidence is described as “unexpected connections that are both riveting and rattling.”[•]

Because probabilities are often stated in news media in terms of odds against an event occurring, students learn to move from probabilities to odds and back. For example, if the odds against a horse winning a race are 4 to 1, the probability that the horse will win is estimated to be $\frac{1}{1+4}$. If the probability that another horse will win is 0.4 then the odds against that horse winning is the probability of not winning, 0.6, to the probability of winning, 0.4, written as 0.6–0.4 or, equivalently, 3–2 or 3 : 2 and read as “3 to 2.” The equivalence of 0.6–0.4 and 3–2 highlights the fact that odds are ratios of numbers, where the numerator and denominators convey meaning. Students learn that the sum of the probabilities of mutually exclusive events occurring cannot exceed 1, but that they sometimes do in media reports where odds and probabilities are approximated for simplicity.

Counting to determine probabilities continues into high school, and student learning is reinforced with models. For example, the birthday problem provides rich learning experiences and shows students some outcomes that are not intuitively obvious. Counting the number of possibilities for n birthdays yields an exponential expression 366^n , and counting of the number of possibilities for n birthdays all to be different yields a permutation P_n^{366} . The quotient is the probability that n randomly chosen people will all have different birthdays, yielding the probability of at least one birthday match among n people. The often surprising result that when $n = 23$ there is approximately a 50–50 chance (probability of 0.5 or 50–50 odds) of having a match. Students learn that the function

$$P(n) = \frac{P_n^{366}}{366^n}$$

models the probability of having no birthday match for n randomly chosen people, and $1 - P(n)$ is the probability of at least one birthday match. The results can be modeled by a spreadsheet revealing the probabilities for $n = 2$ to $n = 367$. Students learn that it requires at least 367 people to have a probability of 1 of a birthday match and also learn about the behavior of technology in that the spreadsheet values for the probability of at least one match become 1 (or at least report as 1) for values of n less than 367. Students calculating $P(n)$ using hand-held calculators learn that for values of n of approximately 40, many hand-held calculators cannot compute the numerators and denominators because of their size. This pro-

- “How surprised should we be?” is attributed to statistician Bradley Efron. “The really unusual day” and other examples are attributed to mathematician Persi Diaconis. See Belkin, 2002, “The Odds of That,” *New York Times Magazine*, <http://www.nytimes.com/2002/08/11/magazine/11COINCIDENCE.html>.

vides an opportunity to learn that rewriting the quotient of the two, too large numbers as the product of a sequence of simpler quotients allows the calculator to compute the sequence of quotients and then take their product. On TI calculators this takes the form

$$\text{Prod} \left(\text{Seq} \left(\frac{x}{366}, x, 366, 366 - n + 1, -1 \right) \right),$$

that is, the product of the sequence

$$\left\{ \frac{366}{366}, \frac{365}{366}, \dots, \frac{366 - n + 1}{366} \right\},$$

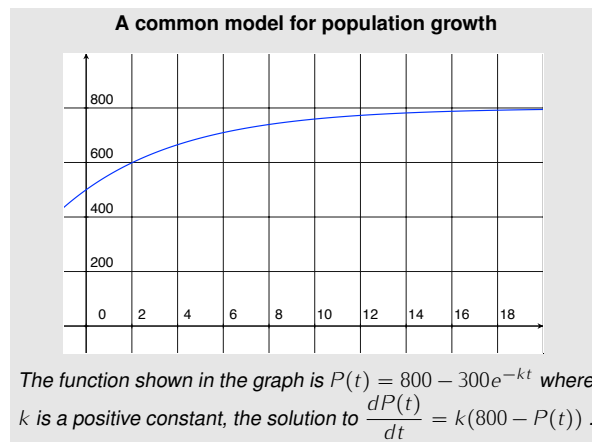
which students learn is a product of probabilities. Students learn that more complex questions can be asked about birthday matches. For example, what is the probability of having exactly one, or exactly two matches of birthdays among n people?

Key features to model

Students learn key features of the graphs of polynomials, rational functions, exponential and logarithmic functions, and modifications such as logistic functions to help in choosing a function that models a real life situation. For example, logistic functions are used in modeling how many students get a certain problem on a test right, and thereby are used in evaluating the difficulty of a problem on a standardized test. A quadratic function might be considered as model of profit from a business if the profit has one maximum (or minimum) over the domain of interest. An exponential function may model a population over some portion of the domain, but circumstances may constrain the growth over other portions. Piecewise functions are considered in situations where the behavior is different over different portions of the domain of interest.

Students learn that real life circumstances such as changes in populations are constrained by various conditions such as available food supply and diseases. They learn that these conditions prevent populations from growing exponentially over long periods of time. A common model for growth of a population P results from a rate of change of P being proportional to the difference between a limiting constant and P , as in Newton's Law of Cooling. This constrained exponential growth results in P being given by the difference between the limiting constant and an exponentially decaying function. For example, the margin shows the graph of a population that is initially 500 and approaches a limiting value of 800. Another common population growth model results from logistic functions where the rate of growth of P is proportional to the product $P(a - P)$ for some constant a .

Students learn to look at key features of the graphs of models of constrained exponential growth (or decay) and logistic functions (intercepts, limiting values, and inflection points) and interpret these key features into the circumstances being modeled.^{F-IF.4}



F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Formulas as models

Formulas are mathematical models of relationships among quantities. Some are statements of laws of nature—e.g., Newton’s Law of Cooling or Ohm’s Law, $V = IR$ —and some are measurements of one quantity in terms of others—e.g., $V = \pi r^2 h$, the volume of a right circular cylinder in terms of its radius and height.^{G-MG.2} Students learn how to manipulate formulas to isolate a quantity of interest. For example, if the question is to what depth will 50 cubic feet of a garden mulch cover a bed of area 20 square feet, then the formula $h = \frac{V}{A}$ where h is the depth, V is the volume, and A is the area, is an appropriate form.^{A-CED.4} If one wants a depth of 6 inches, then the form would be the appropriate for finding how much mulch to buy. Students learn that the shape of the bed (modeled as the base of a cylinder) does not matter, an application of Cavalieri’s Principle;^{G-GMD.1} volume is the product of the area and the height.^{G-GMD.3}

Formulas that are models may sometimes be readily transformed into functions that are models. For example, the formula for the volume of a cylinder can be viewed as giving volume as a function of area of the base and the height, or, rearranging, giving the area of the base as a function of the volume and height. Similarly, Ohm’s Law can be viewed as giving voltage as a function of current and resistance. Newton’s Law of Cooling states that the rate of change of the temperature of a cooling body is directionally proportional to the difference between the temperature of the body and the temperature of the environment, i.e., the ambient temperature.^{F-BF.1b} This is another example of constrained exponential growth (or decay). The solution of this change equation (a differential equation) gives the temperature of the cooling body as a function of time.

In Grade 7, students learned about proportional relationships and constants of proportionality.^{7.RP.2} These surface often in high school modeling. Students learn that many modeling situations begin with a statement like Ohm’s Law or Newton’s Law of Cooling, that is, that a quantity of interest, I , is directly proportional to a quantity, V , and inversely proportional to a quantity, R , i.e. I is given by the product of a constant and $\frac{V}{R}$. Newton’s Law of Cooling is stated as a proportionality giving the rate of change of the temperature at a given moment as a product of a constant and the difference in the temperatures—this can be used in forensic science to estimate the time of death of a murder victim based on the temperature of the body when it is found.

Right triangle and trigonometric models

Students learn that many real world situations can be modeled by right triangles. These include areas of regions that are made up of polygons, indirect measurement problems, and approximations of areas of non-polygonal regions such as circles. Examples are areas

G-MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

A-CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

G-GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

F-BF.1b Write a function that describes a relationship between two quantities.

b Combine standard function types using arithmetic operations.

7.RP.2 Recognize and represent proportional relationships between quantities.

of regular polygons, height of a flag pole, and approximation of the area of a circle by regular polygons. Prior to extending the domains of the trigonometric functions by defining them in terms of arc length on the unit circle, students understand the trigonometric functions as ratios of sides of right triangles. These functions, paired with the Pythagorean Theorem, provide powerful tools for modeling many situations.^{G-SRT.8}

When the domains of the trigonometric functions are extended beyond acute angles,^{F-TF.2} the reasons that these functions are called “circular functions” become clearer. Many situations involving circular motion can be modeled by trigonometric functions. The example below uses trigonometric functions and vector-valued functions. For example, prior to GPSs, this is how a sailor would determine latitude.

Where the Modeling Progression might lead

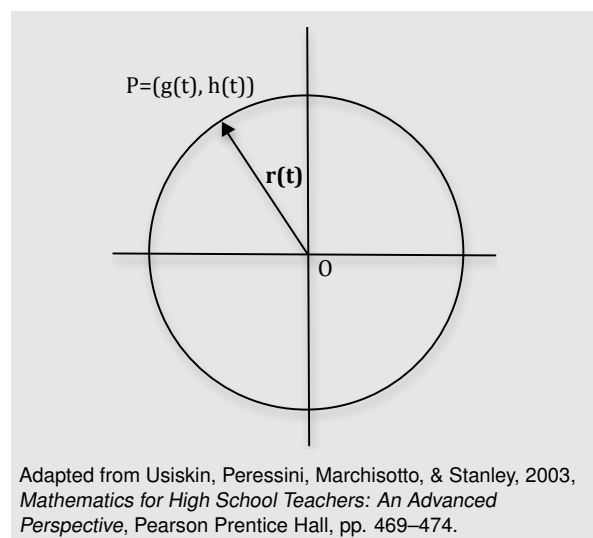
As mentioned earlier, modeling in high school becomes more complex and powerful as more mathematics and statistics are used to describe real life circumstances. As students learn more, they learn to use new concepts to extend simpler models previously studied. Although a high school modeling problem is not likely to incorporate all of high school mathematics, there are models that incorporate many concepts and extend beyond the high school mathematics described in the Standards. The motion of communication satellites around the earth or the motion of an object spinning rapidly in a circle by holding one end of a string with the other attached to the object can be modeled as a point traversing a circle. The object (at point P) is accelerated toward the center (O) of the circular path and the magnitude of the acceleration is constant.

The position vector $\vec{r}(t)$ joining O and P at time t is given by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$ where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ are unit vectors. By considering the geometry and the physics of the situation, one can show that there are functions $g(t)$ and $h(t)$ (twice differentiable, giving the velocity and acceleration vectors of the motion) satisfying the conditions of the model. Noting the similarities of the conditions on g and h to the behavior of the trigonometric functions $\sin(t)$ and $\cos(t)$ one can show that indeed the vector function describes uniform circular motion for an object P on a circle of radius R and a constant magnitude of acceleration.^{F-TF.5}

More examples of modeling in K–12 and beyond appear in the *Guidelines for Assessment and Instruction in Mathematical Modeling Education*. This report, along with information on careers in applied mathematics, is available at the web site of the Society for Industrial and Applied Mathematics, <http://siam.org/reports>.

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.



F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.