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2017 Kansas
Mathematics Standards

Flip Book
4th Grade

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the “flip books.” The “flip books” are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope." 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path, if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size". In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process (Click on photo to view video).

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in Fourth Grade

The Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students grades K-12. Below are a few examples of how these Practices may be integrated into tasks that fourth grade students complete.

Practices	Explanations and Examples
1) Make sense of problems and persevere in solving them.	Mathematically proficient students in grade 4 know that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Fourth graders may use concrete objects or pictures to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" They listen to the strategies of others and will try different approaches. They often will use another method to check their answers.
2) Reason abstractly and quantitatively.	Mathematically proficient students in Grade 4 recognize that a number represents a specific quantity. They extend this understanding from whole numbers to their work with fractions and decimals. This involves two processes - *decontexualizing* and *contextualizing*. Grade 4 students decontextualize by examining a real-world problem and writing and solving equations based on that problem. For example, consider this task: Timothy has $4\frac{1}{8}$ pizzas left over from his soccer party. After giving some pizza to his friend, he has $2\frac{1}{8}$ of pizzas left. How much pizza did Timothy give to his friend? Grade 4 students make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. They need to be able to "show" their thinking using concrete or pictorial representations BEFORE they move to abstract thinking and/or just apply the algorithm without understanding. Further, Grade 4 students write simple expressions, record calculations with numbers, and represent or round numbers using place value concepts.
3) Construct viable arguments and critique the reasoning of others.	In fourth grade mathematically proficient students may construct arguments using concrete referents, such as objects, pictures, and drawings. They explain their thinking and make connections between models and equations. They refine their mathematical communication skills as they participate in mathematical discussions involving questions like "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking.
4) Model with mathematics.	Mathematically proficient students in Grade 4 represent problem situations in various ways, including writing an equation to describe the problem. Students need opportunities to connect the different representations and explain the connections. They should be able to use all of these representations as needed. Students at this level are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multistep problems or problems involving more than one variable. For example, if there is a penny jar that starts with 3 pennies in a jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They use that model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the number sentence $4 \text{ x } 10 + 3$. Then, students in Grade 4 evaluate their results in the context of the situation and reflect on whether the results make sense.

Practices	Explanations and Examples
5) Use appropriate tools strategically.	Mathematically proficient students in Grade 4 consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be helpful. For instance, they may use graph paper or a number line to represent and compare decimals and protractors to measure angles. They use other measurement tools to understand the relative size of units within a system and express measurements given in larger units in terms of smaller units. Proficient students are sufficiently familiar with tools appropriate for 4th grade and areas of content to make sound decisions about when each of the tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations. They choose tools that are relevant and useful to the problem at hand.
6) Attend to precision.	As fourth grader students develop their mathematical communication skills, they try to use clear and precise language in their discussions with others and in their own reasoning. They learn to use mathematical symbols correctly and can describe the meaning of symbols they use and are careful about specifying units of measure. Students in Grade 4 can explain and justify their thinking orally and in writing. They are careful about specifying units of measure and state the meaning of the symbols they choose. For instance, they use appropriate labels when creating a line plot.
7) Look for and make use of structure.	In fourth grade mathematically proficient students look closely to discover a pattern or structure. For instance, students use properties of operations to explain calculations (partial products model). They relate representations of counting problems such as tree diagrams and arrays to the multiplication principal of counting. They generate number or shape patterns that follow a given rule. For example, when calculating 16×9, they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10 + 9) \times 9 = (10 \times 9) + (6 \times 9)$. Another example, students can relate representations of arrays to the multiplication principal of counting. They can also generate number or shape patterns that follow a given rule.
8) Look for and express regularity in repeated reasoning.	Students in fourth grade should notice repetitive actions in computation to make generalizations Students use models to explain calculations and understand how algorithms work. They also use models to examine patterns and generate their own algorithms. For example, students use visual fraction models (pattern blocks, Cuisenaire rods, patty paper, etc.) to show equivalent fractions.

Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others' solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information.

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a______show us that ______may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools~~.~~ (Tools may include~~;~~: concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.).
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
  - a task when there is no need to have an exact answer.
  - a task when there is not enough information to get an exact answer.
  - a task to check if the answer from a calculation is reasonable.

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

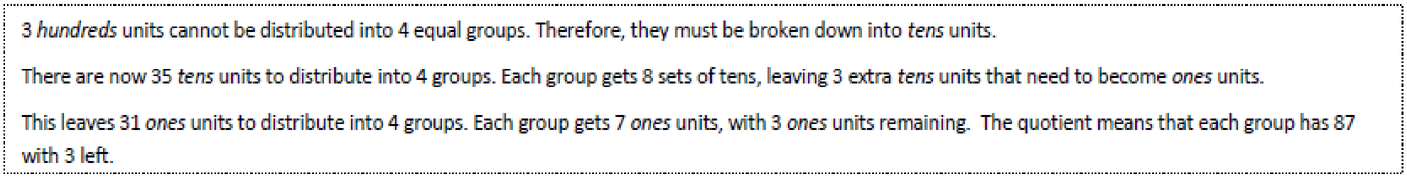
What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem~~.–~~(i.e. – decomposing numbers by place value; working with properties; etc.).
- Asks students to take a complex idea and then identify and use the component parts to solve problems~~.~~ (i.e., Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm). When “unit size” cannot be equally distributed, it is necessary to break down into a smaller “unit size”~~.~~ (example below):

$$\begin{array}{r} 4\overline{)351} \\ \underline{-32} \\ 31 \\ \underline{-28} \\ 3 \end{array}$$



3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.

There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.

This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation~~.~~ i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in 4th Grade

In Grade 4, instructional time should focus on **four** critical areas:

1. **Developing understanding and fluency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends.**
   Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

2. **Developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers.**
   Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., $\frac{35}{10} = \frac{7}{2}$), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.

3. **Refining use of the four operations with whole numbers to solve multistep word problems.**
   Students refine their use of the four operations in order to solve multistep problems efficiently, flexibly and accurately. Students understand that a word problem can be represented with an equation based on the situation, but the solution may use a related equation that is easier to manipulate (e.g., a word problem may be represented with a situation equation such as $345 + ? = 578$; and students understand that even though the word problem is a joining situation, it is easier to solve using a subtraction equation $\{578 - 345 = ?\}$).

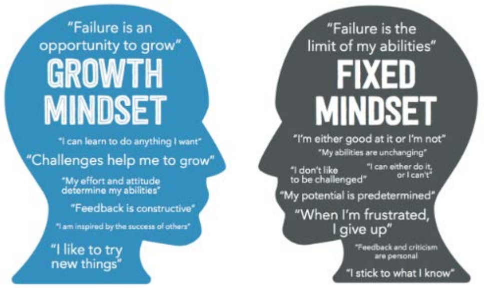
4. **Understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, types of angles, and symmetry.**
   Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset



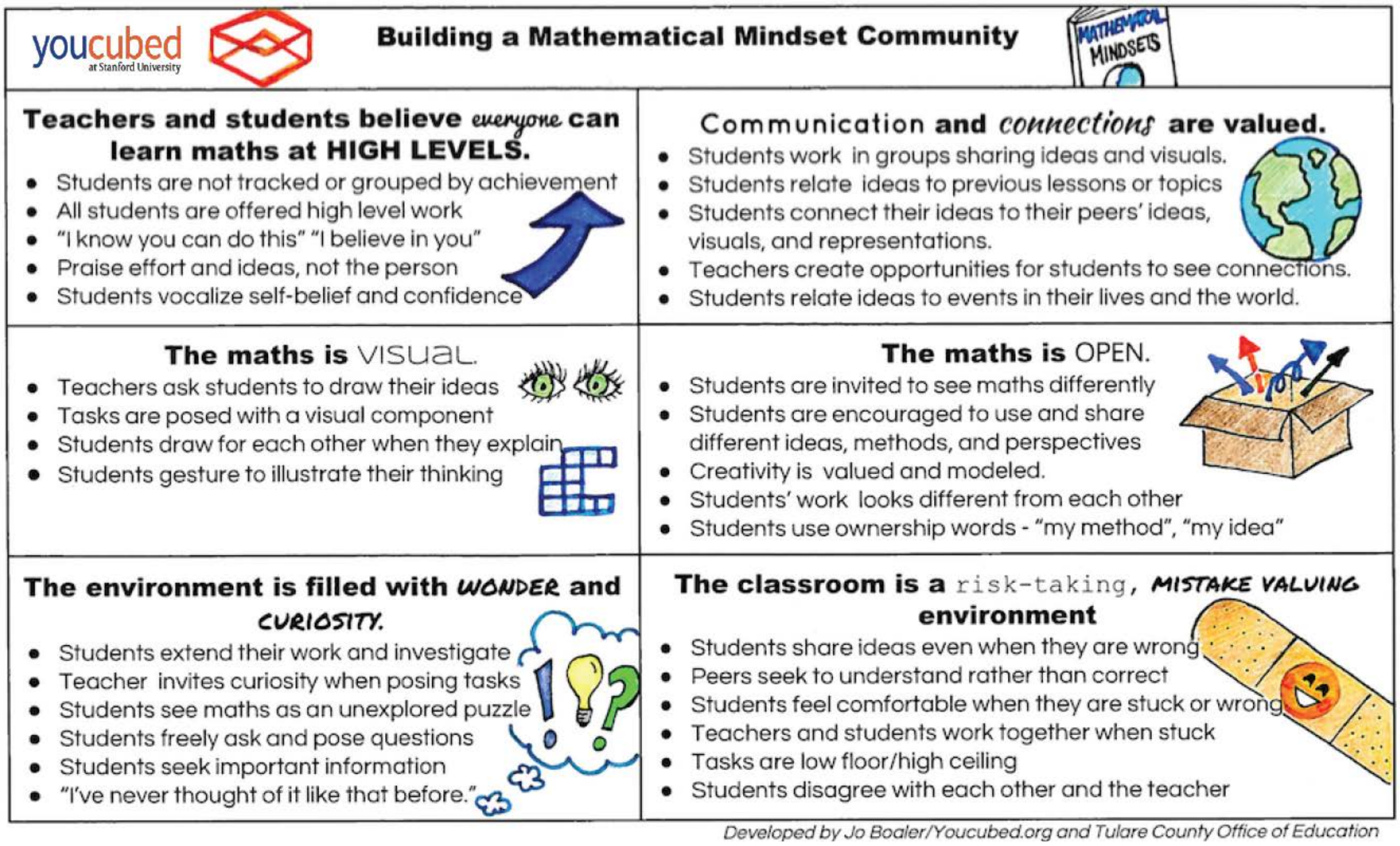
The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.



youcubed at Stanford University

Building a Mathematical Mindset Community

MATHEMATICAL MINDSETS

Teachers and students believe everyone can learn maths at HIGH LEVELS.
- Students are not tracked or grouped by achievement
- All students are offered high level work
- "I know you can do this" "I believe in you"
- Praise effort and ideas, not the person
- Students vocalize self-belief and confidence

Communication and connections are valued.
- Students work in groups sharing ideas and visuals.
- Students relate ideas to previous lessons or topics
- Students connect their ideas to their peers' ideas, visuals, and representations.
- Teachers create opportunities for students to see connections.
- Students relate ideas to events in their lives and the world.

The maths is VISUAL.
- Teachers ask students to draw their ideas
- Tasks are posed with a visual component
- Students draw for each other when they explain
- Students gesture to illustrate their thinking

The maths is OPEN.
- Students are invited to see maths differently
- Students are encouraged to use and share different ideas, methods, and perspectives
- Creativity is valued and modeled.
- Students' work looks different from each other
- Students use ownership words - "my method", "my idea"

The environment is filled with WONDER and CURIOSITY.
- Students extend their work and investigate
- Teacher invites curiosity when posing tasks
- Students see maths as an unexplored puzzle
- Students freely ask and pose questions
- Students seek important information
- "I've never thought of it like that before."

The classroom is a risk-taking, MISTAKE VALUING environment
- Students share ideas even when they are wrong
- Peers seek to understand rather than correct
- Students feel comfortable when they are stuck or wrong
- Teachers and students work together when stuck
- Tasks are low floor/high ceiling
- Students disagree with each other and the teacher

Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education

Grade 4 Content Standards Overview

Operations and Algebraic Thinking (4.OA)

A. Use the four operations with whole numbers to solve problems.
   OA.1 OA.2 OA.3
B. Gain familiarity with factors and multiples.
   OA.4
C. Generate and analyze patterns.
   OA.5

Number and Operations in Base Ten (4.NBT)

A. Generalize place value understanding for multi-digit whole numbers.
   NBT.1 NBT.2 NBT.3
B. Use place value understanding and properties of operations to perform multi-digit arithmetic.
   NBT.4 NBT.5 NBT.6

Number and Operations—Fractions (4.NF)

A. Extend understanding of fraction **equivalence** and ordering.
   NF.1 NF.2
B. Build fractions from unit fractions by applying and extending pervious understandings of operations on whole numbers.
   NF.3 NF.4
C. Understand **decimal notation** for fractions, and compare decimal fractions.
   NF.5 NF.6 NF.7

Measurement and Data (4.MD)

A. Solve problems involving measurement and conversions of measurements from larger units to smaller units.
   MD.1 MD.2 MD.3
B. Represent and interpret data.
   MD.4

Geometry (4.G)

A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.
   G.1 G.2 G.3

Standards for Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

▶ Major Clusters ◆ Supporting Clusters ● Additional Clusters

Domain: Operations and Algebraic Thinking (OA)

▶ *Cluster A:* *Uses the four operations with whole numbers to solve problems.*

Standard: 4.OA.1

Interpret a multiplication equation as a comparison, *(e.g. interpret* $35 = 5 \cdot 7$ *as a statement that 35 is 5 times as many as 7 and 7 times as many as 5.).* Represent verbal statements of multiplicative comparisons as multiplication equations. **(4.OA.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections: (4.OA.1 through 4.OA.3):

This cluster is connected to:

- Represent and solve problems involving multiplication and division (3.OA.A).
- Solve problems involving the four operations, and identify and explain patterns in arithmetic (3.OA.D).

Explanation and Examples:

A *multiplicative comparison* is a situation in which one quantity is multiplied by a specified number to get another quantity (e.g., "*a* is *n* times as much as *b*"). Students should be able to identify and verbalize which quantity is being multiplied and which number tells how many times. They should be given many opportunities to write and identify equations and statements for multiplicative comparisons. It is essential that students are provided many opportunities to solve contextual problems.

Example:

Multiplicative Comparison Idea: Comparison state first and then the equation
5 x 8 = 40.
Sally is five years old. Her mom is eight times older. How old is Sally's Mom?
5 x 5 = 25
Sally has five times as many pencils as Mary. If Sally has 5 pencils, how many does Mary have?

For more detailed information **See Learning Progressions.**

Instructional Strategies: (4.OA.1 through 4.OA.3)

Students need experiences that allow them to connect mathematical statements and number sentences or equations. This allows for an effective transition to formal algebraic concepts. Students should represent an unknown number in a word problem with a symbol. Word problems which require multiplication or division can be solved by using drawings and equations, especially as the students are making sense of the situations.

Resources/Tools:

Table 2 in the Appendix

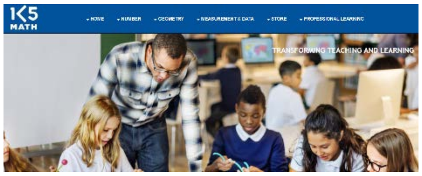
For detailed information see Learning Progression Operations and Algebraic Thinking.

Thinking Blocks video – Multiplicative Comparison Part 1; Multiplicative Comparison Part 2
Thinking Blocks on Math Playground allows students several ways to model problems.

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.OA.A.1
  - Thousands and Millions of Fourth Graders
  - Threatened and Endangered

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.OA.1 to access resources specifically for this standard.



Common Misconceptions:

Students may have "overspecialized" their knowledge of multiplication or division facts and have restricted it to "fact tests" or one particular format. For example, students complete multiplication fact assessments satisfactorily but cannot apply knowledge to problem solving situations.

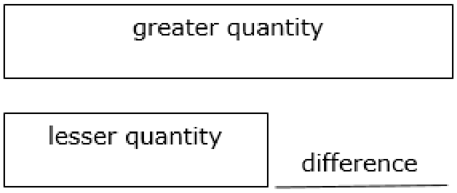
Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A:*** *Use the four operations with whole numbers to solve problems.*

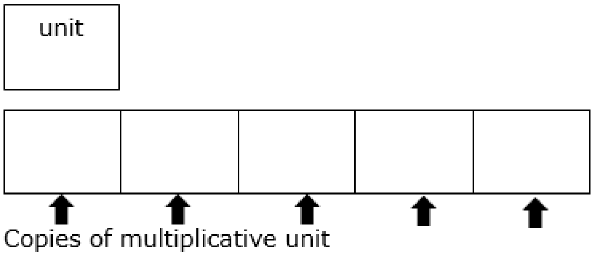
Standard: 4.OA.2

Multiply or divide to solve word problems involving multiplicative comparison, *(e.g. by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison~~.~~).* **(4.OA.2)**

Additive Comparison



Multiplicative Comparison



For Example:
A clown had 20 balloons. He sold some and has 12 left. Each balloon costs $2. How much money did he make?
Situation Equation: $20 - n = 12$
$n \text{ x } \$2 = \square$
Solution Equation: $20 - 12 = n$
$n \text{ x } \$2 = \square$

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.OA.1

Explanation and Examples:

This standard calls for students to translate comparative situations into equations with an unknown and solve.

Students need many opportunities to solve contextual problems. Refer to Table 2 in the Appendix for more specific examples of all types of problems.

Students need to be able to distinguish whether a word problem involves **multiplicative comparison** or **additive comparison** (additive comparison problems are solved using adding and subtracting which was taught in Grades 1 and 2).

Examples:

Unknown Product: A blue scarf costs \$3. A red scarf costs 6 times as much. How much does the red scarf cost? ($3 \times 6 = p$).

Group Size Unknown: A book costs \$18. That is 3 times more than a DVD. How much does a DVD cost? ($18 \div p = 3 \text{ or } 3 \times p = 18$).

Number of Groups Unknown: A red scarf costs \$18. A blue scarf costs \$6. How many times as much does the red scarf cost compared to the blue scarf? ($18 \div 6 = p \text{ or } 6 \times p = 18$).

When distinguishing multiplicative comparison from additive comparison, students should note that:

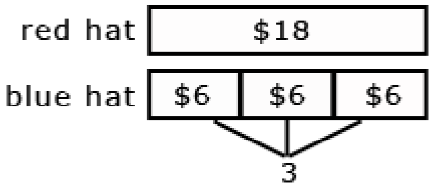
- *Additive comparisons* focus on the difference between two quantities (e.g., Deb has 3 apples and Karen has 5 apples. How many more apples does Karen have?). Remind the students that additive comparison problems are usually asking “*how many more*” or “*how many fewer*” (but may not be exactly those words) between two quantities.
- *Multiplicative comparisons* focus on comparing two quantities by showing that one quantity is a specified number of times larger or smaller than the other (e.g., Deb ran 3 miles. Karen ran 5 times as many miles as Deb. How many miles did Karen run?). Remind students that multiplicative comparisons usually ask “*How many times as much?*” or “*How many times as many?*” but may not use just those phrases exactly.

Students need many opportunities to solve contextual problems. Table 2 in the Appendix, includes the following multiplication problem:

“A blue hat costs \$6. A red hat costs 3 times as much as the blue hat.
How much does the red hat cost?”

When solving this problem, the student should identify \$6 as the quantity that is being multiplied by 3. The student should write the problem using a symbol to represent the unknown.

$$\$6 \times 3 = \square$$



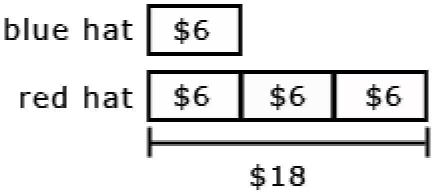
(Question – why can you not multiply money by money? *One of the factors represents the number of groups and so that factor could never be money.*)

Table 2 also includes the following division problem:

A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat?

In solving this problem, the student should identify $18 as the quantity being divided into shares of $6. The student should write the problem using a symbol to represent the unknown.

$$\$18 \div \$6 = \square$$



Instructional Strategies: See 4.OA.1

Tools/Resources:

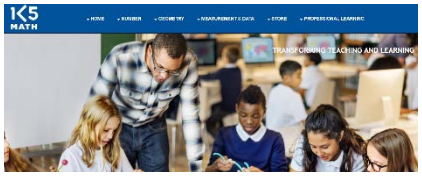
Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.OA.A.2
    - Comparing Money Raised

Thinking Blocks video – Multiplicative Comparison Part 1; Multiplicative Comparison Part 2
Thinking Blocks on Math Playground allows students several ways to model problems.

See also: National Assessment of Educational Progress.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.OA.2 to access resources specifically for this standard.



Common Misconceptions: See 4.OA.1
Students get additive and multiplicative comparisons confused. Have them draw out the situation and explain what is going on in the problem. They should begin to see the patterns within the words of the problems to help them.

Domain: Operations and Algebraic Thinking (OA)

▶ ***Cluster A**: Use the four operations with whole numbers to solve problems.*

Standard: 4.OA.3

Solve multi-step word problem posed with whole numbers and having whole number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using situation equations and/or solution equations with a letter or symbol standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. **(4.OA.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.OA.1

Explanation and Examples:

The focus in this standard is to have students use and discuss various strategies. It refers to **estimation** strategies, these should include using compatible numbers (numbers that sum to 10 or 100) and rounding. Problems should be structured so that all acceptable estimation strategies will arrive at a reasonable answer. Students need many opportunities solving multistep story problems using all four operations.

Example:

On a vacation, your family travels 267 miles on the first day, 194 miles on the second day and 34 miles on the third day. How many total miles did they travel? Some typical estimation strategies for this problem:

Student 1	Student 2	Student 3
I first thought about 267 and 34. I noticed that their sum is about 300. Then I knew that 194 is close to 200. When I put 300 and 200 together, I get 500.	I first thought about 194. It is really close to 200. I also have 2 hundreds in 267. That gives me a total of 4 hundreds. Then I have 67 in 267 and the 34. When I put 67 and 34 together that is really close to 100. When I add that hundred to the 4 hundreds that I already had, I end up with 500.	I rounded 267 to 300. I rounded 194 to 200. I rounded 34 to 30. When I added 300, 200 and 30, I know my answer will be about 530.

The assessment of estimation strategies should only have one reasonable answer (500 or 530), or a range (between 500 and 550). Problems should be structured so that all acceptable estimation strategies will arrive at a reasonable answer.

Example 2:

Your class is collecting bottled water for a service project. The goal is to collect 300 bottles of water. On the first day, Creighton brings in 3 packs with 6 bottles in each container. Susan wheels in 6 packs with 6 bottles in each container. About how many bottles of water still need to be collected?

Student 1	Student 2
First, I multiplied 3 packs by 6 bottles in each pack which equals 18 bottles. Then I multiplied 6 packs by 6 bottles in each pack which is 36 bottles. I know 18 plus 36 is about 50 bottles. I'm trying to get to 300 bottles. 50 plus another 50 is 100 bottles. Then I need 2 more hundreds. So we still need 250 bottles.	First, I multiplied 3 packs by 6 bottles in each pack which equals 18 bottles. Then I multiplied 6 packs by 6 bottles in each pack which is 36 bottles. I know 18 is about 20 and 36 is about 40 bottles. 40 + 20 = 60. 300 - 60 = 240, so we need about 240 more bottles.

This standard also references <u>interpreting remainders</u>. Remainders should be put into context for interpretation. Ways to address remainders:

- Remainder as a "leftover"
- Partitioned into fractions or decimals
- Discarded leaving only the whole number answer
- Increase the whole number answer by one
- Round to the nearest whole number for an approximate result

Example:

Have the students write different **word** problems using $44 \div 6 =$ ___ where the answers are best represented as:

- Problem A: 7
- Problem B: $7r2$
- Problem C: 8
- Problem D: 7 or 8
- Problem E: $7\frac{2}{6}$

Possible solutions:

Problem A: 7.

Mary had 44 pencils. Six pencils fit into each of her pencil pouches. How many pouches did she completely fill?

$44 \div 6 = p; p = 7r2$. *Mary can fill 7 pouches completely.*

Problem B: $7r2$.

Mary had 44 pencils. Six pencils fit into each of her pencil pouches. How many pouches could she fill and how many pencils would she have left?

$44 \div 6 = p; p = 7r2$; *Mary can fill 7 pouches and have 2 left over.*

Problem C: 8.

Mary had 44 pencils. Six pencils fit into each of her pencil pouches. What would the fewest number of pouches she would need in order to hold all of her pencils?

$44 \div 6 = p; p = 7r2$; *Mary needs 8 pouches to hold all of her pencils.*

Problem D: 7 or 8.

Mary had 44 pencils. She divided them equally among her friends before giving one of the leftovers to each of her friends. How many pencils could her friends have received?

$44 \div 6 = p; p = 7r2;$ *Some of her friends received 7 pencils. Two friends received 8 pencils.*

Problem E: $7\frac{2}{6}$

Mary had 44 pencils and put six pencils in each pouch. What fraction represents the number of pouches that Mary filled?

$$44 \div 6 = p; p = 7\frac{2}{6}.$$

Example of interpreting the remainder to round up:

There are 128 students going on a field trip. If each bus held 30 students, how many buses are needed? *(*$128 \div 30 = b;$ $b = 4r8;$ They will need 5 buses because 4 busses cannot hold all of the students*).*

Students need to realize in some problems, such as the one above, an extra bus is needed for the 8 students that are left over.

Examples to discuss:

Kim is making candy bags. There will be 5 pieces of candy in each bag. She had 53 pieces of candy. She ate 14 pieces of candy. How many candy bags can Kim make now?

(53 – 14) ÷ 5 = 7*r*4

(7 bags with 4 leftover)

There are 29 students in one class and 28 students in another class going on a field trip. Each car can hold 5 students. What is the fewest number of cars that are needed to get all the students to the field trip?

$29 + 28 = 57; 57 \div 5 = 11r2.$

(12 cars will be needed because 11 cars will only hold 55 students.)

Estimation skills include identifying when estimation is appropriate, determining the level of accuracy needed, selecting the appropriate method of estimation, and verifying solutions or determining the reasonableness of situations using various estimation strategies.

Estimation strategies include, but are not limited to:

- front-end estimation with adjustment (use the highest place value and estimate from that place value – the front of the number, then make adjustments to the estimate by taking into account the amounts from the other place values that were initially ignored).
- rounding and adjusting (students round down or round up and then adjust their estimate depending on how much the rounding affected the original values).
- using friendly or compatible numbers (students seek to fit numbers together - e.g., rounding to factors and grouping numbers together that have round sums like 100 or 1000).
- using benchmark numbers with fractions that are easy to compute (students select close whole numbers of fractions or decimals to determine an estimate).

Students need many opportunities solving multistep story problems using all four operations and ALL *situations* found in Tables 1 and 2 in the Appendix.

Instructional Strategies:

Students need to solve word problems involving multiplicative comparison (product unknown, partition unknown) using multiplication or division as shown in Table 2 in the Appendix (Common Multiplication and Division Situations). They should use drawings or equations with a symbol for the unknown number to represent the problem.

Present multistep word problems with whole numbers and whole-number answers using the four operations. Students should know which operations are needed to solve the problem. Drawing pictures or using models will help students understand what the problem is asking. They should check the reasonableness of their answer using estimation strategies and mental computation.

Offer word problems to students with the numbers covered up or replaced with symbols or icons. Ask the students to write the equation or the number sentence to show the situation of the problem. The equation or the number sentence can be considered the "answer" to the problem or you can replace the symbols or icons with numbers and have the students solve the problem after discussion of the situation and processing all the information.
Example: There are □ students in one class and ◆ students in another class going on a field trip. Each car can hold ▯ students. What is the fewest number of cars that are needed to get all the students to the field trip?

Tools/Resources:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.OA.A.3
    - Karl's Garden
    - Carnival Tickets

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.OA.3 to access resources specifically for this standard.

Thinking Blocks on Math Playground allows students several ways to model word problems involving all operations.

Greg Tang's Word Problem Generator – allows you to select all the various situation subtypes but they are only 1-step word problems. Good for reviewing the situations but you need to move quickly to multi-step word problems so your 4th grade students become comfortable with these types of problems.

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- A Squadron of Bugs

NRICH: enriching mathematics

- Remainders
- Reasoned Rounding

Examples of multistep word problems can be accessed from the released questions on the NAEP (National Assessment of Educational Progress) Assessment.

For example, a constructed response question from the 2007 Grade 4 NAEP assessment reads, *“Five classes are going on a bus trip and each class has 21 students. If each bus holds only 40 students, how many buses are needed for the trip?”*

Common Misconceptions:

Students apply a procedure that results in remainders that are expressed as *r* for ALL situations, even for those in which the result does not make sense. For example when a student is asked to solve the following problem, the student responds to the problem—there are 32 students in a class canoe trip. They plan to have 3 students in each canoe. How many canoes will they need so that everyone can participate? And the student answers “$10r2$ canoes”. What does that mean? Does the student understand how to interpret the remainder?

Domain: Operations and Algebraic Thinking (OA)

◆ ***Cluster B:*** *Gain familiarity with factors and multiples.*

Standard: 4.OA.4

Find all factor pairs for a whole number in the range 1 to 100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1 to 100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1 to 100 is prime or composite. **(4.OA.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Understand properties of multiplication and the relationship between multiplication and division (3.OA.5 through 3.OA.6).
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition (3.MD.8).
- The concepts of prime, factor and multiple are important in the study of relationships found among the natural numbers. Compute fluently with multi-digit numbers and find common factors and multiples (6.NS.4).

Explanation and Examples:

Mathematically proficient students communicate precisely by engaging in discussion about their reasoning using appropriate mathematical language. The following are terms students should learn to use with increasing precision: **multiplication/multiply, division/divide, factor pairs, factor, multiple, prime, composite**

This standard requires students to demonstrate understanding of factors and multiples of whole numbers. This standard also refers to prime and composite numbers. **Prime numbers** have exactly two factors, the number one and the number itself. (For example; the number 17 has the factors of 1 and 17 only so it is a prime number.) Composite numbers have more than two factors. (For example, 12 has the factors of 1, 2, 3, 4, 6 and 12. Since this number has more factors than 1 and 12, it is a composite number.)

A common misconception is that the number 1 is prime, when in fact; it is neither prime nor composite. (See this explanation from Wolfram MathWorld if you need more specific information.)

Another common misconception is that all prime numbers are odd numbers. This is not true, since the number 2 has only two factors (1 and 2) and is also an even number.

Prime vs. Composite:

A prime number is a number greater than 1 that has only 2 factors, 1 and itself. Composite numbers have more than 2 factors.

Students investigate whether numbers are prime or composite by:

- building rectangles (arrays) with the given area and finding which numbers have more than two rectangles (e.g. 7 can be made into only 2 rectangles, 1 x 7 and 7 x 1, therefore it is a prime number).
- finding factors of the number.

Students should understand the process of finding factor pairs so they can do this for any number 1 through 100.

Example:
Factor pairs for 96: 1 and 96, 2 and 48, 3 and 32, 4 and 24, 6 and 16, 8 and 12.

Multiples can be thought of as the result of skip counting by each of the factors. When skip counting, students should be able to identify the number of factors counted e.g., 5, 10, 15, 20 (there are 4 fives in 20).

Example:
Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Multiples of each factor for 24: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24
3, 6, 9, 12, 15, 18, 21, 24
4, 8, 12, 16, 20, 24
8, 16, 24
12, 24
24

To determine if a number between 1 and 100 is a multiple of a given one-digit number, some helpful hints include the following:

- All even numbers are multiples of 2.
- All even numbers that can be halved twice (with a whole number result) are multiples of 4.
- All numbers ending in 0 or 5 are multiples of 5.
- If the sum of the digits is 3, 6, or 9 then it is a multiple of 3.

Instructional Strategies:

Students need to develop an understanding of the concepts of number theory, such as **prime numbers** and **composite numbers**. This includes the relationship of factors and multiples. Multiplication and division are used to develop concepts of factors and multiples. Division problems resulting in remainders are used as counter-examples of factors.

Review vocabulary so that students have an understanding of terms such as *factor*, *product*, *multiples*, and *odd* and *even* numbers.

Students need to develop strategies for determining if a number is **prime** or **composite;** in other words, if a number has a whole number factor that is not one or itself. Starting with a number chart of 1 to 20, use multiples of prime numbers to eliminate later numbers in the chart. For example, 2 is prime but 4, 6, 8, 10, 12 . . . are composite.

After working with the numbers 1 to 20, consider using a hundreds chart and have the students color code multiples of numbers. The color will help students see emerging patterns which they can discuss.

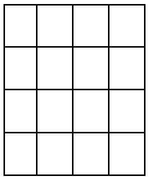
Encourage the development of rules that can be used to aid in the determination of **composite** numbers. For example, *other than 2, if a number ends in an even number (0, 2, 4, 6 and 8), it is a composite number*?

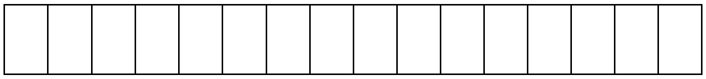
Using area models will also assist students in analyzing numbers and arriving at an understanding of whether a number is **prime** or **composite**. Have students construct rectangles with an area equal to a given number. They should see an association between the number of rectangles and the given number for the area as to whether this number is a prime or composite number. Numbers with only 2 possible area models are **prime** numbers.

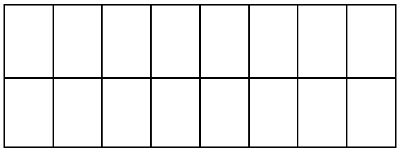
Definitions of prime and composite numbers should not be provided, but determined after many strategies have been used in finding all possible factors of a number.

Provide students with counters to find the factors of numbers. Have them find ways to separate the counters into equal subsets. For example, have them find several factors of 10, 14, 25 or 32, and write multiplication expressions for the numbers.

Another way to find the factor of a number is to use arrays from square tiles or drawn on grid papers. Have students build rectangles that have the given number of squares. For example if you have 16 squares:







The idea that a product of any two whole numbers is a common multiple of those two numbers is a difficult concept to understand. For example, 5 x 8 is 40; the table below shows the multiples of each factor.

5	10	15	20	25	30	25	40	45
8	16	24	32	40	48	56	64	72

Ask students what they notice about the number 40 in each set of multiples; 40 is the 8th multiple of 5, and the 5th multiple of 8. Why?

Knowing how to determine factors and multiples is the foundation for finding common multiples and factors in Grade 6, BUT students in the 4th grade are not required to find common factors or common multiples.

Resources/Tools

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.OA.B
    - Identifying Multiples
    - Numbers in a Multiplication Table
    - Multiples of 3, 6, and 7
- 4.OA.B.4
    - The Locker Game

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- "The Factor Game " - This engages students in a friendly contest in which winning strategies involve distinguishing between numbers with many factors and numbers with few factors. Students are then guided through an analysis of game strategies and introduced to the definitions of prime and composite numbers.
- *"The Product Game– Classifying Numbers "* - Students construct Venn diagrams to show the relationships between the factors or products of two or more numbers in the game.
- "Multiplication: It's in The Cards" - Patterns with Products.

National Center for Education Statistics: NAEP Questions Tool

National Library of Virtual Manipulatives.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.OA.4 to access resources specifically for this standard.

Factors and Multiples for Two & Factors and Multiples Puzzle from NRICH website.

Common Misconceptions:

When listing multiples of numbers, students may not list the number itself. Emphasize that the smallest multiple is the number itself. Also, having students write multiples of a number by consecutive factors beginning with one can clear up this misconception.

Some students may think that larger numbers have more factors. Having students share all factor pairs and how they found them will clear up this misconception.

Domain: Operations and Algebraic Thinking (OA)

● ***Cluster C**: Generate and analyze patterns.*

Standard: 4.OA.5

Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers.* Explain informally why the numbers will continue to alternate in this way. **(4.OA.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

- Solve problems involving the four operations, and identify and explain patterns in arithmetic. (3.OA.8 & 3.OA.9).

Explanation and Examples:

Mathematically proficient students communicate precisely by engaging in discussion about their reasoning using appropriate mathematical language. The terms students should learn to use with increasing precision with this cluster are: **pattern (number or shape patterns), pattern rule.**

Patterns involving numbers or symbols either repeat or grow. Students need multiple opportunities creating and extending number and shape patterns. Numerical patterns allow students to reinforce facts and develop fluency with operations.

Patterns and rules are related. A pattern is a sequence that repeats the same process over and over. A rule dictates what that process will look like. Students investigate different patterns to find rules, identify features in the patterns, and justify the reason for those features.

Examples:

Pattern	Rule	Feature(s)
3, 8, 13, 18, 23, 28…	Start with 3, add 5	The numbers alternately end with a 3 or 8
5, 10, 15, 20…	Start with 5, add 5	The numbers are multiples of 5 and end with either 0 or 5. The numbers that end with 5 are products of 5 and an odd number. The numbers that end in 0 are products of 5 and an even number.

After students have identified rules and features from patterns, they need to generate a numerical or shape pattern from a given rule.

Example:

Rule: Starting at 1, create a pattern that multiplies each number by 3. Stop when you have 6 numbers generated for your pattern.

> Students write 1, 3, 9, 27, 81, 243. Students notice that all the numbers are odd and that the sums of the digits of the numbers with 2 or more digits are each 9. Some students might investigate this beyond six numbers.

In this standard, students **describe** features of an arithmetic number pattern or shape pattern by identifying the rule, and features that are not explicit in the rule. A t-chart is a tool to help students see number patterns.

Example:

There are 4 beans in the jar. Each day 3 beans are added. How many beans are in the jar for each of the first 5 days?

Day	Operation	Beans
0	3x0+4	4
1	3x1+4	7
2	3x2+4	10
3	3x3+4	13
4	3x4+4	16
5	3x5+4	19

Instructional Strategies:

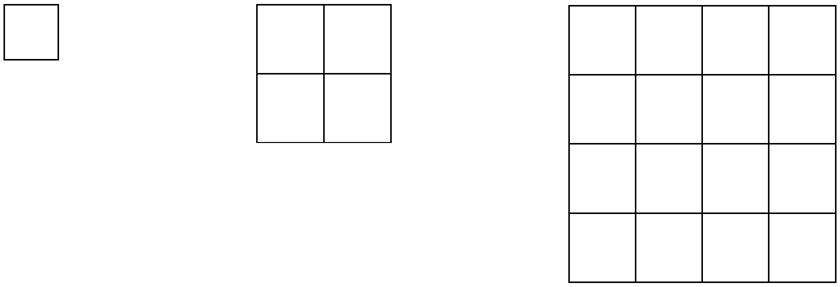
In order for students to be successful later in the formal study of algebra, their algebraic thinking needs to be developed. Understanding patterns is fundamental to algebraic thinking. Students have experience in identifying arithmetic patterns, especially those included in addition and multiplication tables. Utilizing contexts that are familiar to students is helpful in developing students' algebraic thinking.

Students should generate numerical or geometric patterns that follow a given rule. They should look for relationships in the patterns and be able to describe and make **generalizations**.

As students generate numeric patterns for rules, they should be able to "undo" the pattern to determine if the rule works with all of the numbers generated. For example, given the rule, "Add 4" starting with the number 1, the pattern 1, 5, 9, 13, 17, … is generated. In analyzing the pattern, students need to determine how to get from one term to the next term. Teachers can ask students, "How is a number in the sequence related to the one that came before it?", and "If they started at the end of the pattern, will this relationship be the same?" Students can use this type of questioning in analyzing numbers patterns to determine the rule.

Students should also determine if there are other relationships in the patterns. In the numeric pattern generated above, students should observe that the numbers are all odd numbers. Then discuss why this is happening~~?~~. Can they explain their reasoning and their thoughts to others?

Provide patterns that involve shapes so that students can determine the rule for the pattern. For example, examine the pattern with the geometric figures below:



What is happening? Can you explain?

Tools/Resources:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

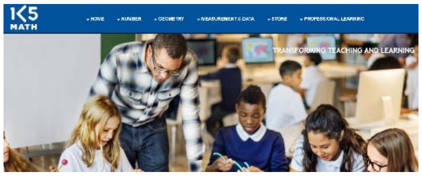
- 4.OA.C
    - 4Multiples of 3, 6, and 7
- 4.OA.C.5
    - Double Plus One
    - Multiples of nine

"Snake Patterns –s-s-s", PBS Teachers. Students will use given rules to generate several stages of a pattern and will be able to predict the outcome for any stage.

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- "Patterns that Grow – Growing Patterns" - Students use numbers to make growing patterns. They create, analyze, and describe growing patterns and then record them. They also analyze a special growing pattern called Pascal's triangle.
- "Patterns that Grow – Exploring Other Number Patterns" - Students analyze numeric patterns, including Fibonacci numbers. They also describe numeric patterns and then record them in table form.
- "Patterns that Grow – Looking Back and Moving Forward " - In this final lesson of the unit, students use logical thinking to create, identify, extend, and translate patterns. They make patterns with numbers and shapes and explore patterns in a variety of mathematical contexts.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.OA.5 to access resources specifically for this standard.



Sticky Triangles, Counter Patterns, & Exploring Number Patterns You Make from NRICH mathematics.

Common Misconceptions:

Students think that results are random. There is no pattern. Another common misconception when students are working with repeating patterns is that they will often repeat what is given rather than looking at what "chunks" or part of the pattern is actually being repeated. Example: Given the pattern 6,9,12,6,9,12,6,9,… If the student is asked "*what is the next number in the pattern*", they may respond with "6" because they are returning to the beginning of the given pattern and repeat it from there. Students should be encouraged to look for the repeating "unit".

Domain: Number and Operations in Base Ten (NBT)

▶ *Cluster A:* *Generalize place value understanding for multi-digit whole numbers*.

Standard: 4.NBT.1

Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that* $700 \div 70 = 10$ *by applying concepts of place value and division.* **(4.NBT.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: (4.NBT.1 through 4.NBT.3)

This cluster is connected to:

- A strong foundation in whole-number place value and rounding is critical for the expansion to decimal place value and decimal rounding.
- Understand place value (2.NBT1 through 2.NBT.4).
- Use place value understanding and properties of operations to perform multi-digit arithmetic (3.NBT.1).

Explanation and Examples:

Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

This standard expects students to extend their understanding of place value related to multiplying and dividing by multiples of 10. In this standard, students should reason about the magnitude of the digits in a number. Students should be given many opportunities to reason and analyze the relationships of numbers they are working with.

Example:

Explain how the digit 2 in the number 582 is similar to and different from the digit 2 in the number 528?

Students should be able to explain that the digit is the same but the **value** is very different in each number.

Instructional Strategies: (4.NBT.1)

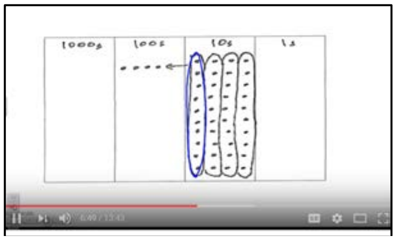
Students should be familiar with and use place value as they work with numbers. Some activities that will help students develop understanding of this standard are:

- Investigate the product of 10 and any number, then justify why the number now has a 0 “at the end”. ($7 \times 10 = 70$ because 70 represents 7 tens and no ones, $10 \times 35 = 350$ because the 3 in 350 represents 3 hundreds, which is 10 times as much as 3 tens, and the 5 represents 5 tens, which is 10 times as much as 5 ones.) While students can easily see the pattern of adding a 0 at the end of a number when multiplying by 10, they need to be able to justify why this works using place value language.
- Investigate the pattern, 6, 60, 600, 6,000, 60,000, 600,000. What needs to happen to 600 to make 6000? (Adding a zero on the end is NOT an explanation that is based in place value understanding).

Resources/Tools

For detailed information see Learning Progression Number and Operations in Base Ten

Ten Times Greater video by Emily Jemison helps to show and explain place value for 4th grade.



Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NBT.A
  - What's My Number?
- 4.NBT.A.1
  - Threatened and Endangered
  - Thousands and Millions of Fourth Graders

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NBT.1 to access resources specifically for this standard.

In this resource from NRICH, this standard is explained with some resources attached – look most particularly at #4 and #5.

- Supporting the Development of Place Value

Common Misconceptions: (4.NBT.1)

Students do not understand that when they “add” a 0 to the end of a number that they have created a “short cut” of multiplying by 10. Challenge them to explain. Students should always explain and justify what they are doing so they can understand the mathematics behind their actions.

Students do not understand that when the digit moves to the left that it has increased a place value which is the same thing as multiplying by 10. There needs to be much work and discussion about why this is happening.

Domain: Number and Operations in Base Ten (NBT)

▶ ***Cluster A:*** *Generalize place value understanding for multi-digit whole numbers.*

Standard: 4.NBT.2

Read and write multi-digit whole numbers using base-ten numerals, number names, expanded form, and unit form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, <, =, and ≠ symbols to record the results of comparisons. *(Note: Students should demonstrate understanding and application of place value decomposition. For example, 127 can be 1 hundred, 2 tens, 7 ones OR 12 tens, 7 ones. Refer to 2.NBT.1)* **(4.NBT.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.NBT.1

Explanation and Examples:

This standard expects students to read numbers appropriately and to write numbers in all forms and have flexibility with the different forms. For the number 285 the following forms are: traditional expanded form is 285 = 200 + 80 + 5; number name is two hundred eighty-five; unit form is 2 hundreds, 8 tens, and 5 ones OR 2 hundreds + 8 tens + 5 ones. Students should also have opportunities to explore the idea that 285 could also be 28 tens plus 5 ones or 1 hundred, 18 tens, and 5 ones (this concept started in 2nd grade with numbers up to 100).

Students should **use place value to compare numbers**. For example, in comparing 34,570 and 34,192, a student might say, both numbers have the same value of 10,000s and the same value of 1000s, however, the value in the 100s place is different so that is where the comparison of the two numbers would be determined.

NOTE: Students should be instructed in the meaning of the ≠ symbol along with the others listed. Math education experts recommend students use = and ≠ first. Once students have determined that numbers are not equal, then they can determine "how" they are not equal – which one is < and which one is >. If students cannot determine if amounts are ≠ or = then they will struggle with < or >.

Instructional Strategies: See 4.NBT.1

Provide multiple opportunities in the classroom setting and use real-world context for students to read and write multi-digit whole numbers.

Students need to have several opportunities to **compare** numbers in various situations. They need to compare numbers with the same number of digits, (e.g., compare the three numbers: 453 698 215); numbers that have the same number in the leading digit position, (e.g., compare the three numbers: 45 495 41,223); and numbers that have different numbers of digits and different leading digits, (e.g., compare the four numbers: 312 95 5245 10,002).

Students should be creating numbers that meet specific criteria. For example, provide students with cards numbered 0 through 9. Ask students to select 4 to 6 cards, then using all the cards make the largest number possible with all cards, OR the smallest number possible, OR the closest number to 5000, OR a number that is greater than 5000, OR a number that is less than 5000, etc. Then discussions with the students about the numbers will solidify their understanding.

Resources/Tools:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NBT.A.2
    - Ordering 4-digit numbers

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NBT.2 to access resources specifically for this standard.

Common Misconceptions: See 4.NBT.1

There are several misconceptions students may have about writing numerals from verbal descriptions. Numbers like one thousand often do not cause a problem; however a number like one thousand two can cause problems for students. Many students will understand the 1000 and the 2 but then instead of placing the 2 in the ones place, students will write the numbers as they hear them, 10002, not understanding that this number represents more than 1002.

Students often assume that the first digit of a multi-digit number indicates the "greatness" of a number. The assumption is made that 954 is greater than 1002 because students are focusing on the first digit instead of the number as a whole.

Domain: Number and Operations in Base Ten (NBT)

▶ ***Cluster A**: Generalize place value understanding for multi-digit whole numbers.*

Standard: 4.NBT.3

Use place value understanding to round multi-digit whole numbers to any place. **(4.NBT.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.NBT.1

Explanation and Examples:

This standard refers to place value understanding, which extends **beyond** an algorithm or procedure for rounding. The expectation is that students have a deep understanding of place value and number sense and can explain and reason about the answers they get when they round. Students should have numerous experiences using a number line and a hundred chart as tools to support their work with rounding.

Example:

Your class is collecting bottled water for a service project. The goal is to collect 300 bottles of water. On the first day, Max brings in 3 packs with 6 bottles in each pack. Sarah wheels in 6 packs with 6 bottles in each. About how many bottles of water still need to be collected?

Student 1	Student 2
First, I multiplied 3 and 6 which equals 18 bottles. Then I multiplied 6 and 6 which is 36 bottles. I know 18 plus 36 is about 50 bottles. I'm trying to get to 300, so 50 plus another 50 is 100 bottles. Then I need 2 more hundreds. So we still need about 250 bottles.	First, I multiplied 3 and 6 which equals 18 bottles. Then I multiplied 6 and 6 which is 36 bottles. I know 18 is about 20 and 36 is about 40, so 40+20=60 bottles. 300-60 = 240 bottles, so we need about 240 more bottles.

Example:

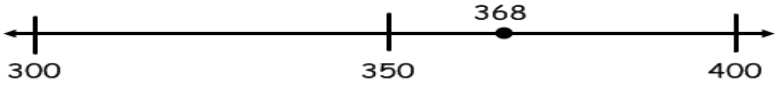
On a vacation, your family travels 267 miles on the first day, 194 miles on the second day, and 34 miles on the third day. About how many total miles did they travel? Some typical estimation strategies for this problem:

Student 1	Student 2	Student 3
I first thought about 267 and 34. I noticed that their sum is about 300 miles. Then I knew that 194 is close to 200. When I put 300 and 200 together, I get about 500 miles.	I first thought about 194. It is really close to 200. I also have 2 hundreds in 267. That gives me a total of 4 hundreds. Then I have 67 in 267 and the 34. When I put 67 and 34 together that is really close to 100. When I add that hundred to the 4 hundreds that I already had, I end up with 500. So about 500 miles.	I rounded 267 to 300. I rounded 194 to 200. I rounded 34 to 30. When I added 300, 200 and 30, I know my answer will be about 530 miles.

Students will also need to round numbers not in context but it is recommended to start with a context so students understand why we need to know how to round efficiently and appropriately.

Example:
Round 368 to the nearest hundred.

Using a number line:
Find the nearest hundreds before and after 368.
Draw a number line and place those hundreds at each end of the line. Divide the number line at the midpoint to show the benchmark number.
Determine whether 368 is closer to 300 or 400. Since 368 is past 350 and closer to 400, this number should be rounded to 400.



Another example:
Round 76,398 to the nearest 1000.

- Rounding to the nearest 1000 means I need to look at the thousands place and greater to determine the thousands before and after this number. (76,398 – I know that 76 thousands is before this number and 77 thousands is after this number {76,000 and 77,000}).
- The halfway point between these two numbers is 76,500.
- I see that 76,398 is before 76,500 so the rounded number would be **76,000**.

Instructional Strategies: See 4.NBT.1
In Grade 4, rounding should not be new, and students need to build on the Grade 3 skill of rounding to the nearest 10 or 100 to include larger numbers and place value. What is new for Grade 4 is rounding to digits other than the leading digit, e.g., round 23,960 to the nearest hundred. This requires greater sophistication than rounding to the nearest ten thousand because the digit in the hundreds place represents 900 and when rounded it becomes 1,000, not just zero.

Students should also begin to develop some efficient rules for rounding building from the basic strategy of - “Is 48 closer to 40 or 50?” Since 48 is only 2 away from 50 and 8 away from 40, 48 would round to 50. Number Lines are effective tools for this type of thinking. Students need to **generalize** the rule for much larger numbers and rounding to values that are not the leading digit.

Tools/Resources:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NBT.A.3
    - Rounding to the Nearest 1000
    - Rounding to the Nearest 100 and 1000

Common Misconceptions: See 4.NBT.1

Students who do not have a firm understanding of place value will struggle with rounding. You may need to diagnose place value gaps before proceeding with this skill.

Trying to do this skill without visual tools leads for misconceptions about the magnitude of numbers.

Domain: Number and Operations in Base Ten (NBT)

▶ *Cluster B*: *Use place value understanding and properties of operations to perform multi-digit arithmetic*.

Standard: 4.NBT.4

Fluently (efficiently, accurately, and flexibly) add and subtract multi-digit whole numbers using an efficient algorithm (including, but not limited to: traditional, partial-sums, etc.), based on place value understanding and the properties of operations. **(4.NBT.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (4.NBT.4 through 4.NBT.6)

This Cluster is connected to:

- Use place value understanding and properties of operations to perform multi-digit arithmetic~~.~~ (3.NBT.2 through 3.NBT.3).
- Use the four operations with whole numbers to solve problems (4.OA.3).
- Generalize place value understanding for multi-digit whole numbers (4.NBT.1 through 4.NBT.2).

Explanation and Examples:

Students build on their understanding of addition and subtraction, their use of place value, and their flexibility with multiple strategies to make sense efficient algorithms. They continue to use place value and the properties of operations in describing and justifying the processes they use to add and subtract.

This standard refers to **fluency, which means accuracy and efficiency (using a reasonable amount of steps and time), and flexibility (using a variety of strategies which incorporate the properties of operations and the decomposition and composition of numbers.).** Kansas State Department of Education White Paper on Fluency

Fourth grade is the level in which students are expected to be proficient at using the traditional algorithm to add and subtract. However, other previously learned strategies are still expected to be used based on the problem. Students are expected to evaluate all problems and determine which strategy, or strategies, are most efficient to solve them. If your students are instantly regrouping for a problem such as 100 – 94, then they are not efficient or flexible. Students should be able to tell you that the solution is 6 without hesitation. When asked to explain how they know, your students should be able to refer to the problem as a whole and share that 94 is only 6 away from 100, or talk about the number line or hundred chart that was visualized in their head. Blind application of any type of algorithm without thinking about the problem is not efficient nor flexible.

When students begin using the traditional algorithm their explanation may be quite lengthy. After much practice with using place value to justify their steps, they will develop fluency with the algorithm. Students should be able to explain why the algorithm works.

Example:

$$\begin{array}{r} 3892 \\ \underline{+1567} \end{array}$$

Student explanation for the traditional algorithm (all traditional algorithms begin with the least place value) could be:

1. Two plus seven is nine so I write that in the ones place.
2. Nine tens plus six tens is 15 tens. I am going to write down 5 tens and think of the 10 tens as one more one hundred (notates with a 1 to represent the hundred above or below the hundreds column).
3. Eight hundreds plus five hundreds plus the extra hundred from adding the tens is 14 hundreds. I am going to write the 4 hundreds and think of the 10 hundreds as one more thousand (notates with a 1 above or below the thousands column).
4. Three thousands plus one thousand plus the extra thousand from the hundreds is 5 thousand. So my answer is 5459.

$$\begin{array}{r} 3892 \\ \underline{+1567} \\ 1\,1 \\ 5459 \end{array}$$

Student explanation for the partial-sums algorithm (all partial algorithms begin with the greatest place value) could be:

1. Three thousand and one thousand are four thousand so I write that below the problem.
2. Eight hundreds and five hundreds are 13 hundreds which I will write below the 4000. 13 hundreds is written as 1300.
3. Nine tens and six tens are 15 tens. 15 tens is written as 150 below the 1300.
4. Two and seven are nine which is written below the 150.
5. All the partial sums are added together to get 5459.

$$\begin{array}{r} 3892 \\ \underline{+1567} \\ 4000 \\ 1300 \\ 150 \\ \underline{9} \\ 5459 \end{array}$$

$$\begin{array}{r} 3546 \\ \underline{-\ 928} \end{array}$$

Student explanation for the traditional algorithm (all traditional algorithms begin with the least place value) could be:

1. In this problem there are not enough ones to take 8 ones from the 6 ones, so I have to regroup my tens and ones by taking one ten from the 4 tens, decompose it into 10 ones and put it in the ones place. Now I have 3 tens and 16 ones. (To show this on paper, mark through the 4 in the tens place and notate with a 3, then write a 1 next to the 6 in the ones column to represent the 16 ones.)
2. Now I have enough ones so I can take eight ones from sixteen ones to be left with 8 ones. (Write an 8 in the ones column below the problem.)
3. Move to the tens column. Three tens minus two tens is 1 ten. (Write a 1 in the tens column below the problem.)
4. Move to the hundreds column. There are not enough hundreds to take 9 hundreds from 5 hundreds, so I need to regroup my thousands and hundreds by taking one thousand from the 3 thousands, decompose it into 10 hundreds and put it into the hundreds place. Now I have 2 thousands and 15 hundreds. (Mark through the 3

$$\begin{array}{r} \overset{2}{\cancel{3}}{}^{1}5\overset{3}{\cancel{4}}{}^{1}6 \\ \underline{-\ 928} \\ 2618 \end{array}$$

thousands and notate with a 2 above it. Write a 1 next to the 5 in the hundreds column to represent 15 hundreds.)

5. Now I have enough hundreds so I can take 9 hundreds from 15 hundreds to get 6 hundreds. (Write a 6 in the hundreds column below the problem.)~~.~~
6. I have 2 thousands left since I do not have to take away any thousands. (Writes 2 in the thousands place below the problem.)~~.~~ My solution is 2618.

Student explanation for a variation of the traditional algorithm using place value could be:

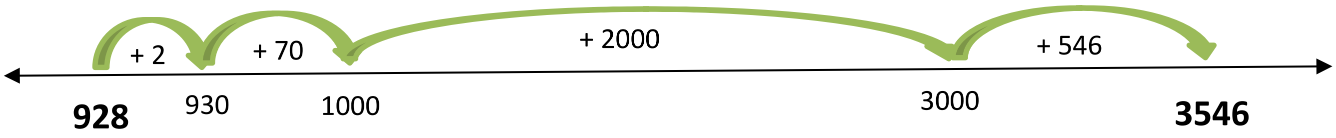
1. There are not enough ones to take 8 ones from the 6 ones, so I have to regroup my tens and ones by taking one ten from the 354 tens, decompose it into 10 ones and put it in the ones place. Now I have 353 tens and 16 ones. (To show this on paper, mark through the 354 tens and notate with 353 tens, then write a 1 next to the 6 in the ones column to represent the 16 ones.)
2. Now I have enough ones so I can take eight ones from sixteen ones to be left with 8 ones. (Write an 8 in the ones column below the problem.)
3. Move to the tens column. Three tens minus two tens is 1 ten. (Write a 1 in the tens column below the problem.)
4. Move to the hundreds column. There are 35 hundreds so I can take 9 hundreds from that to have 26 hundreds left. (Write 26 in the hundreds column below the problem.) My solution is 2618.

$$\begin{array}{r} \overset{353}{\cancel{354}}\,{}^{1}6 \\ \underline{-\ 928} \\ 2618 \end{array}$$

Student explanation for the open number line strategy could be:

1. I can make jumps to show the difference between 928 and 3546.
2. My first jump is 2 to 930.
3. Then I jump 70 to get to 1000.
4. I jump 2000 to get to 3000.
5. Then I jump 546 to get to 3546.
6. I add all my jumps together to get 2618.

$$\begin{array}{r} 3546 \\ \underline{-\ 928} \end{array}$$



Note: Students should know that it is mathematically possible to subtract a larger number from a smaller number but that their work with whole numbers does not allow this as the difference would result in a negative number.

Instructional Strategies: (4.NBT.4 through 4.NBT.6)

A crucial theme in multi-digit arithmetic is encouraging students to develop *strategies* that they understand, can explain, and can think about using place value and the properties of operations, rather than merely following a sequence of directions, rules or procedures that they don't understand. It is important for students to have seen and to use a variety of strategies and materials to broaden and deepen their understanding of place value before they are required to use traditional algorithms. The goal is for students to *understand* all the steps in an algorithm, and be able to explain the meaning of each digit throughout the process.

For example, a 1 can represent one, one ten, or one hundred, and so on. For multi-digit addition and subtraction in Grade 4, the goal is also **fluency**, which means students must be able to carry out the calculations efficiently, accurately, and flexibly.

Start with students' understanding of a certain strategy, and then make intentional, clear-cut connections to the any algorithm. This allows the student to gain understanding of the algorithm rather than just memorizing certain steps to follow.

Sometimes students benefit from 'being the teacher' to an imaginary (or an actual) student who is having difficulties applying any algorithm in addition and subtraction situations. To promote understanding, use examples of student work that has been done incorrectly and ask students to provide feedback about the student work.

It is very important for students to talk through their understanding of connections between different strategies and algorithms. Give students many opportunities to talk with classmates about how they could explain algorithms. Think-Pair-Share is a great structure for all students when sharing their thinking.

Tools/Resources

See EngageNY Math Module for 4.NBT.4

Georgia Department of Education:

- See: "Grocery Shopping - This task provides students with the opportunity to apply estimation strategies and an understanding of how estimation can be used as a real life application. For this activity, it is expected that students have been introduced to rounding as a process for estimating.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NBT.4 to access resources specifically for this standard.

Video for Partial Sums Addition Strategy.

Video for Subtraction Using an Open Number Line.

Common Misconceptions: (4.NBT.4 through 4.NBT.6)

Often students get confused when asked to 'carry' or 'borrow' since they are really regrouping the different place values. Make sure you use appropriate vocabulary when talking with your students.

Often do not notice the need for regrouping in subtraction and just take the smaller digit from the larger one. Emphasize place value and the meaning of each of the digits.

Some students struggle with being accurate in representing the specific place values on their paper. It can be helpful to have them write their calculations on grid paper or lined notebook paper (with the lines running vertically) to assist the students with lining up the place values accurately.

Domain: Number and Operations in Base Ten (NBT)

▶ ***Cluster B***: *Use place value understanding and properties of operations to perform multi-digit arithmetic*.

Standard: 4.NBT.5

Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. **(4.NBT.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections:

- Represent and solve problems involving multiplication and division (3.OA.A) & understand properties of multiplication and the relationship between multiplication and division (3.OA.B).
- Multiply one-digit whole numbers by multiples of 10 in the range 10 to 90 using strategies based on place value and the properties of operations (3.NBT.3).

Explanation and Examples:

Students who develop flexibility in breaking numbers apart (decomposing numbers) have a better understanding of the importance of place value and the distributive property in multi-digit multiplication.

Students use base ten blocks, area models, partitioning, compensation strategies, etc. when multiplying whole numbers and use words and diagrams to explain their thinking. They use the terms factor and product when communicating their reasoning. Multiple strategies enable students to develop fluency with multiplication and transfer that understanding to division. (Use of the traditional algorithm for multiplication and understanding why it works, is not an expectation until the 5th grade.)~~.~~

This standard expects students to multiply numbers using a variety of strategies based on understanding. They are expected to develop their accuracy, efficiency and flexibility of the various strategies and algorithms.

Example of different strategies students could use:

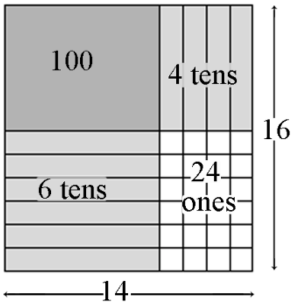
There are 25 dozen cookies in the bakery. What is the total number of cookies at the bakery?

Student 1	**Student 2**	**Student 3**
25 x 12 I broke 12 into 10 and 2. So 25 x 10 = 250 and 25 x 2 = 50. Now I add the partial products so 250 + 50 = 300.	25 x 12 I broke 25 up into 5 groups of 5.Then 5 x 12 = 60. I have 5 groups of 5 so 60 x 5 = 300.	25 x 12 I doubled 25 to 50 and halved 12 to get 6. Now I have 50 x 6 = 300. (*Some students may continue to double and halve to get 100 x 3*.)

Use of place value and the distributive property are applied in the scaffolded examples below.

- To illustrate 154 x 6 students use base 10 blocks or use drawings to show 154 six times. Seeing 154 six times will lead them to understand the distributive property, 154 X 6 = (100 + 50 + 4) x 6 = (100 x 6) + (50 X 6) + (4 X 6) = 600 + 300 + 24 = 924.
- The area model shows the partial products algorithm in visual form.

14 x 16 = 224



100 + 40 + 60 + 24 = 224

Using the area model, students first verbalize their understanding:

- 10 x 10 is 100
- 4 x 10 is 40
- 10 x 6 is 60, and
- 4 x 6 is 24.

They use different strategies to record this type of thinking.

- Students explain the two strategies below with base 10 blocks, drawings, or numbers.

25
X 24
400 (20 x 20)
100 (20 x 5)
80 (4 x 20)
20 (4 x 5)
600

25
X 24
500 (20 x 25)
100 (4 x 25)
600

- **Open Area Model:** This model can be used after students have a good understanding of the area model strategy shown in #2 above.

	20	5	
20	400	100	500
4	80	20	100
	480 +	120	600

Example:

What would an open area model of 74 X 38 look like?

	70	4
30	$70 \times 30 = 2{,}100$	$4 \times 30 = 120$
8	$70 \times 8 = 560$	$4 \times 8 = 32$

$$2{,}000 = 560 + 1{,}200 + 32 = 2{,}812$$

Instructional Strategies: See 4.NBT.4

Use the examples shown above in the explanation to plan instructional lessons that will allow students to develop an understanding of the strategies and algorithms based in place value and the properties of operations.

Tools/Resources

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NBT.B.5
    - Thousands and Millions of Fourth Graders

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NBT.5 to access resources specifically for this standard.

Georgia Department of Education:

- ”Using Arrows to Multiply Bigger Numbers” - In this task students demonstrate how to multiply two-digit numbers using arrays. Students will be given a multiplication problem with a two-digit number by a two-digit number. They will use graph paper to solve the problem by breaking it down into partial products (smaller arrays to find the answer).

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- Factorize

NRICH mathematics shows various different multiplication strategies.

Open Area Model (or Open Array Strategy)

For detailed information see Progressions for the Common Core State Standards in Mathematics: K-5, Number and Operations in Base Ten

Common Misconceptions: See 4.NBT.4

Students that believe they have already “learned” the traditional algorithm frequently don’t want to try any other strategies. They claim they are confused or don’t understand or it is too hard. This just shows that these students don’t really have a deep understanding. They are not being flexible or efficient in their thinking. In order for students to truly understand multiplication they need to be able to explain the process using multiple strategies and multiple tools.

Domain: Number and Operations in Base Ten (NBT)

▶ ***Cluster B:*** *<u>Use place value understanding and properties of operations to perform multi-digit arithmetic</u>.*

Standard: 4.NBT 6

Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. **(4.NBT.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.NBT.4

Explanation and Examples:

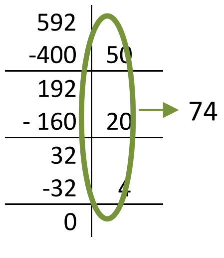
In fourth grade, students build on their third grade work with division within 100. Students need opportunities to develop their understandings by using problems in context as well as out of context. Introducing division in context will allow students to visualize and make sense of the process of division.

Examples:

A 4th grade teacher bought 4 plastic boxes to hold all of the extra pencils she bought at the school supply sale. She has 260 pencils. She wants to put the pencils in the boxes so they are evenly divided. How many pencils does she need to place into each box?

- **Using Base 10 Blocks**: Students build 260 with base 10 blocks and distribute them into 4 equal groups. Some students may need to trade the 2 hundreds for tens but others may easily recognize that 200 divided by 4 is 50.
- **Using Place Value thinking to Decompose:** 260 ÷ 4 = (200 ÷ 4) + (60 ÷ 4)
- **Using Multiplication:** 4 x 50 = 200, 4 x 10 = 40, 4 x 5 = 20; 50 + 10 + 5 = 65; so 260 ÷ 4 = 65

This standard expects students to explore division through multiple strategies based on place value and the properties of operations. Here are some examples of students thinking about how to solve 592 ÷ 8:

Student 1	**Student 2**	**Student 3**
592 ÷ 8 = ? There are seventy 8's in 560. 592 - 560 = 32 There are four 8's in the remaining 32. So 70 + 4 = **74**.	592 ÷ 8 = ? I know that ten 8's is 80. If I take out fifty 8's that is 400. 592 - 400 = 192 I can take out twenty more 8's which is 160 So 192 - 160 = 32 I can take four 8s out of 32 so now I have none left I took out 50, then 20 more, then 4 more, so that makes **74**. 	592 ÷ 8 = ? I want to get to 592 8 x 25 = 200 8 x 25 = 200 8 x 25 = 200 200 + 200 + 200 = 600 600 - 8 = 592 I had 75 groups of 8 and took one group away, so there are **74** groups.

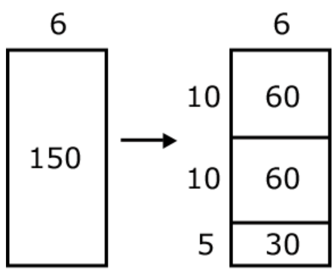
Example:

Using an Open Array or Area Model

After developing an understanding of using arrays to divide, students begin to use a more abstract model for division. This model connects to a recording process that will be formalized in the 5th grade. (**Note**: When using the "house" symbol for division, it actually represents the corner of an array or area model. So when you are showing the division of 150 by 6, the 150 represents the number within the array or area model and the number to the left represents the number of rows. So the number you are solving for at the top will provide the number of columns - $\begin{array}{r} ? \\ 6\overline{)150} \end{array}$).

Open Array Method for Division

150 ÷ 6



Students make a rectangle and write 6 on one of its sides. They express their understanding that they need to think of the rectangle as representing a total of 150.

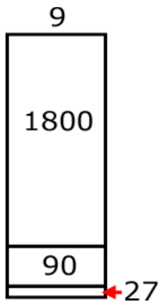
1. Students think, 6 times what number is a number close to 150? They recognize that 6 x 10 is 60 so they record 10 as a factor and partition the rectangle into 2 rectangles and label the area aligned to the factor of 10 with 60. They express that they have only used 60 of the 150 so they have 90 left.
2. Recognizing that there is another 60 in what is left they repeat the process above. They express that they have used 120 of the 150 so they have 30 left.
3. Knowing that 6 x 5 is 30. They write 30 in the bottom area of the rectangle and record 5 as a factor.
4. Students express their calculations in various ways:
    a.

$$\begin{array}{l} 150 \\ \underline{-60(6\times 10)} \\ 90 \\ \underline{-60(6\times 10)} \\ 30 \\ \underline{-30(6\times 5)} \\ 0 \end{array} \qquad 150 \div 6 = 10 + 10 + 5 = 25$$

    b. $150 \div 6 = (60 \div 6) + (60 \div 6) + (30 \div 6) = 10 + 10 + 5 = 25$

Another example using the open array method:

$$1917 \div 9$$



A student's description of his or her thinking may be:

I need to find out how many 9s are in 1917. I know that 200 x 9 is 1800. So if I use 1800 of the 1917, I have 117 left. I know that 9 x 10 is 90. So if I have 10 more 9s, I will have 27 left. I can make 3 more 9s. I have 200 nines, 10 nines and 3 nines. So I made 213 nines.

$$1917 \div 9 = 213$$

Instructional Strategies: See 4.NBT.4
Use the examples shown above in the explanation to plan instructional lessons that will allow students to develop an understanding of the strategies and algorithms based in place value and the properties of operations.

Tools/Resources
For detailed information see Progressions for the Common Core State Standards in Mathematics: K-5, Number and Operations in Base Ten

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NBT.B.6
    - Mental Division Strategy

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NBT.6 to access resources specifically for this standard.

Video for the Open Array Model for Division.
Video for the Open Array Model having Remainders.

Article from NRICH mathematics explaining how to use arrays in multiplication and division.

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- Quotient Café.

Common Misconceptions: See 4.NBT.4

Often students have “learned” the traditional algorithm for division and tend to look at the digits within the number as single digits instead of thinking about the place value of each digit or even thinking about the number as a whole. When asked if their solution is reasonable, students do not understand what is reasonable because they are unable to estimate since they don’t see the number in its entirety, but rather, as individual digits. Students must have a solid understanding about place value and the properties of operations in order to make sense of division.

Domain: Number and Operations – Fractions (NF)

▶ ***Cluster A**: Extend understanding of fraction equivalence and ordering.*
(Limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, 100.)

Standard: 4.NF.1

Explain why a fraction $\frac{a}{b}$ is equivalent to a fraction $\frac{(n\cdot a)}{(n\cdot b)}$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. **(4.NF.1)** (Number and Operations—Fractions Progression 3–5 Pg. 6)

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (4.NF.1 through 4.NBT.2)

This cluster is connected to:

- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers (4.NF.B).
- Develop understanding of fractions as numbers (3.NF.A).

Explanation and Examples:

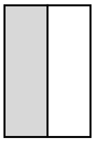
This standard refers to visual fraction models. This includes area models, linear models (number lines), and also collection/set models. The set or collection models may be new to fourth graders since they were not expected in third grade.

This standard extends the work in third grade by using additional denominators (5, 10, 12, and 100). Students should use all visual models, and can also use digital applets, to generate equivalent fractions.

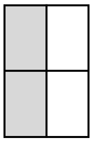
Example:

All the area models below show $\frac{1}{2}$. The second model shows $\frac{2}{4}$ but also shows that $\frac{1}{2}$ and $\frac{2}{4}$ are **equivalent fractions** because their areas are **equivalent**. When a horizontal line is drawn through the center of the model, the number of equal parts doubles and the size of the parts is halved.

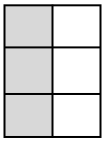
Students will begin to notice connections between the models and their fractions in the way both the parts and wholes are counted and begin to generate a rule for writing equivalent fractions. (*The rule is not expected in 4th grade*.)



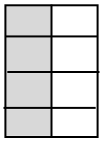
$\frac{1}{2}$



$\frac{2}{4} = \frac{2\times1}{2\times2}$



$\frac{3}{6} = \frac{3\times1}{3\times2}$



$\frac{4}{8} = \frac{4\times1}{4\times2}$

Instructional Strategies: **(4.NF.1)**

Students' initial formal experience with fractions began in Grade 3 (some informal foundations were laid in the geometry standards in the grades previous to 3rd grade). They used models (area/region models and linear/measurement models) for making sense of unit fractions, locating fractions and understanding their magnitude, and recognizing and generating equivalent fractions. They also used these models to explain and justify their reasoning about fractions when comparing them.

Third grade students were expected to understand the importance of knowing the whole, the unit fraction, and how all fractions are created from equal-sized parts.

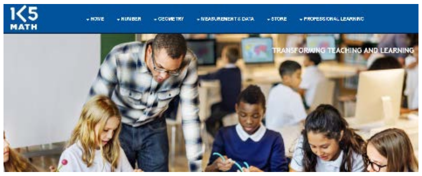
Tools/Resources

For Additional Information See Progressions for the Common Core State Standards in Mathematics: 3-5 Number and Operations-Fractions

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.A
    - Money in the piggy bank
    - Running Laps
- 4.NF.A.1
    - Explaining Fraction Equivalence with Pictures
    - Fractions and Rectangles

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.1 to access resources specifically for this standard.



This technology connection is an activity to create equivalent fractions by dividing shapes and matching them to number line locations.

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- Equivalent Fractions

Videos: Equivalent Fractions on a Number Line, Equivalent Fractions on a Number Line from iLearn, Bar Model with Equivalent Fractions

Common Misconceptions: (4.NF.1 through 4.NF.2)

Students think that when generating equivalent fractions they need to multiply or divide either the numerator or denominator, such as, changing $\frac{1}{2}$ to sixths. Often this is not efficient. Reasoning about their size using benchmark fractions is more efficient and solidifies students' understanding about the size of the fraction.

You know students don't understand the reasoning behind generating equivalent fractions because they will multiply just the denominator by 3 to get $\frac{1}{6}$, instead of multiplying the numerator by 3 also. Their focus is only on the multiple of the denominator, not the "whole fraction". It is important that students use a fraction in the form of **one** such as $\frac{3}{3}$ so that an equivalent fraction is created.

Domain: Number and Operations- Fractions (NF)

▶ ***Cluster A:*** *Extend understanding of fraction equivalence and ordering.*
(Limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, 100.)

Standard: 4.NF.2

Compare two fractions with different numerators and different denominators, *(e.g. by creating common numerators or denominators, or by comparing to a benchmark fraction such as $\frac{1}{2}$).* Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with relational symbols >, <, =, or ≠, and justify the conclusions, *(e.g. by using visual fraction models.).* **(4.NF.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.NF.1

Explanation and Examples:

Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., $\frac{15}{9} = \frac{5}{3}$), and they develop methods for generating and recognizing equivalent fractions and can represent equivalent fractions concretely and/or pictorially.

This standard expects students to compare fractions by creating visual fraction models or finding common denominators or numerators. **Students' experiences should focus on visual fraction models rather than algorithms**. Students should learn to draw fraction models to help them compare and use reasoning skills based on fraction benchmarks.

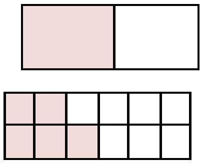
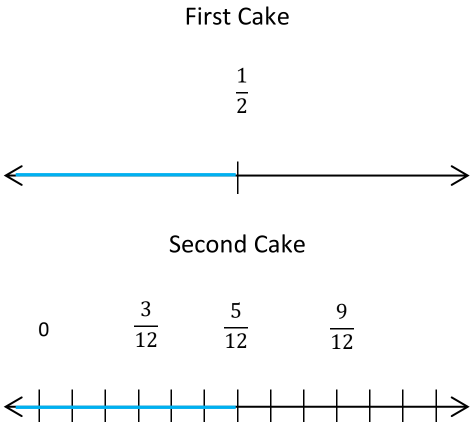
Students must also recognize that they must consider the size of the whole when comparing fractions (i.e., $\frac{1}{2}\ and\ \frac{1}{8}$ of two medium pizzas is very different from $\frac{1}{2}$ of one medium and $\frac{1}{8}$ of one large). Record the results of comparisons with symbols >, <, =, or ≠, and justify the conclusions, e.g., by using a visual fraction model.

Example:
Use patterns blocks.

1. If a red trapezoid is one whole, which block shows $\frac{1}{3}$?
2. If the blue rhombus is $\frac{1}{3}$, which block shows one whole?
3. If the red trapezoid is one whole, which block shows $\frac{2}{3}$?

Example:

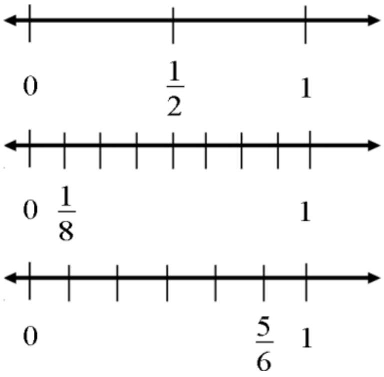
There are two cakes on the counter that are the same size. The first cake has $\frac{1}{2}$ of it left. The second cake has $\frac{5}{12}$ left. Which cake has more left?

Student 1	**Student 2**	**Student 3:**
Area model: The first cake has more left over. The second cake has $\frac{5}{12}$ left which is smaller than $\frac{1}{2}$. 	**Linear/Number Line model:** 	I know that $\frac{6}{12}$ equals $\frac{1}{2}$. Therefore, the second cake which has $\frac{7}{12}$ left is greater than $\frac{1}{2}$. **Benchmark fractions** include common fractions between 0 and 1 such as halves, thirds, fourths, fifths, sixths, eighths, tenths, twelfths, and hundredths.

Fractions can be compared using benchmarks, common denominators, or common numerators. Symbols used to describe comparisons include >, <, =, or ≠.

It is important that students explain the relationship between the numerator and the denominator using Benchmark Fractions. See examples below:

Fractions may be compared using $\frac{1}{2}$ as a benchmark.



Possible student thinking by using benchmarks:

$\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when 1 whole is cut into 2 pieces.

Possible student thinking by creating common denominators:

$$\frac{5}{6} > \frac{1}{2} \text{ because } \frac{3}{6} = \frac{1}{2} \text{ and } \frac{5}{6} > \frac{3}{6}$$

Fractions with common denominators may be compared using the numerators as a guide.

$$\frac{2}{6} < \frac{3}{6} < \frac{5}{6}$$

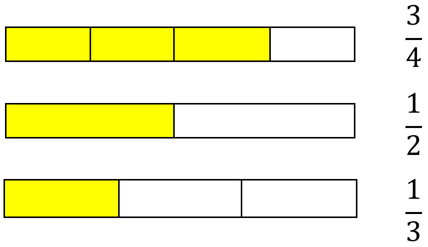
Fractions with common numerators may be compared and ordered using the denominators as a guide.

$$\frac{1}{10} < \frac{3}{8} < \frac{3}{4}$$

Instructional Strategies:

Students in the fourth grade are expected to extend their understanding of unit fractions to compare two fractions with different numerators and different denominators. Students should use models (the same models that are described above with the addition of the set model) to compare two fractions with different denominators by creating common denominators or common numerators. The models should be the same (region, set, number line, etc.) so that the models represent the <u>same whole</u>. The models should eventually be represented in drawings as students move from the concrete to the pictorial stage of learning.

Students should use **benchmark fractions** such as $\frac{1}{2}$ to compare two fractions and justify that comparison by explaining their reasoning using the benchmark fraction. The result of the comparisons should be recorded using >, <, ≠ or = symbols.



Example of Conceptual Reasoning by using a Benchmark Fraction:

How does $\frac{5}{8}$ compare to $\frac{1}{2}$?

"I know the two fractions are not equal. I know that $\frac{5}{8}$ is a little bit more ($\frac{1}{8}$ more) than the benchmark fraction of $\frac{1}{2}$, because $\frac{4}{8}$ is equal to $\frac{1}{2}$, so I would place $\frac{5}{8}$ just to the right of $\frac{1}{2}$ on the number line. So $\frac{5}{8}$ is greater than $\frac{1}{2}$".

Tools/Resources: See 4.NF.1

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.A.2
    - Doubling Numerators and Denominators
    - Listing fractions in increasing size
    - Using Benchmarks to Compare Fractions

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.2 to access resources specifically for this standard.

Common Misconceptions: See 4.NF.1

Students think that when generating equivalent fractions they need to multiply or divide either the numerator or denominator, such as, changing $\frac{1}{2}$ to sixths. Often this is not efficient. Reasoning about their size using benchmark fractions is more efficient and solidifies students' understanding about the size of the fraction

Domain: Number and Operations – Fractions (NF)

▶ ***Cluster B:*** *Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.*
(Limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, 100.)

Standard: 4.NF.3

Understand a fraction $\frac{a}{b}$ with $a > 1$ as a sum of fractions $\frac{1}{b}$.

4.NF.3a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. **(4.NF.3a)**

4.NF.3b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, *e.g. by using a visual fraction model.* **(4.NF.3b)**

Examples: $\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$; $\frac{3}{8} = \frac{1}{8} + \frac{2}{8}$; $2\frac{1}{8} = 1 + 1 + \frac{1}{8} = \frac{8}{8} + \frac{8}{8} + \frac{1}{8}$.

4.NF.3c. Add and subtract mixed numbers with like denominators, *e.g. by replacing each mixed number with an equivalent fraction (simplest form is not an expectation), and/or by using properties of operations and the relationship between addition and subtraction.* **(4.NF.3c)**

4.NF.3d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, *e.g. by using visual fraction models and equations to represent the problem.* **(4.NF.3d)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (4.NF.3 through 4.NF.4)

This cluster is connected to:

- Fourth Grade - Developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers.
- Represent and interpret data (4.MD.B)

Explanation and Examples:

A fraction with a numerator of one is called a **unit fraction.** When students investigate fractions other than unit fractions, such as $\frac{2}{3}$, they should be able to decompose the non-unit fraction into a combination of several unit fractions.

Example:

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3}$$

Being able to visualize this decomposition into unit fractions helps students when they are adding or subtracting fractions.

Students will need multiple opportunities to work with mixed numbers to be able to decompose them in more than one way. Students may use visual models to help develop this understanding.

Example:

$$\underbrace{1\frac{1}{4}}-\frac{3}{4}=\square$$

$$\begin{tikzcd} \boxed{\frac{4}{4}+\frac{1}{4}=\frac{5}{4}} \arrow[r] & \boxed{\frac{5}{4}-\frac{3}{4}=\frac{2}{4}\ or\ \frac{1}{2}} \end{tikzcd}$$

Example of word problem:

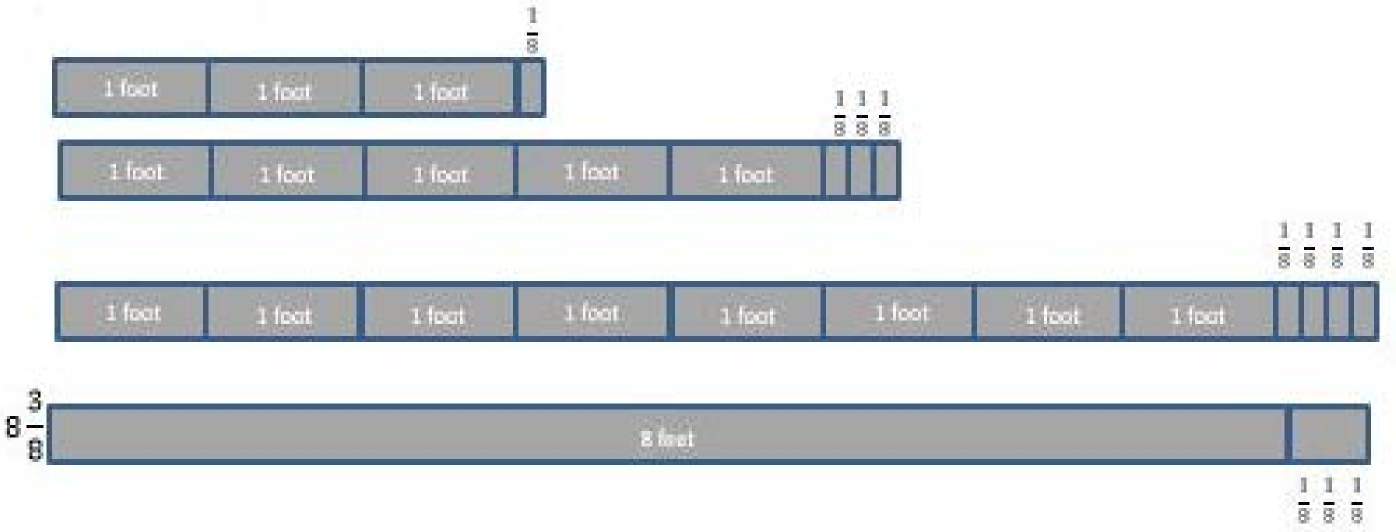
> Mary and Lacey decide to share a pizza. Mary ate $\frac{3}{6}$ and Lacey ate $\frac{2}{6}$ of the pizza. How much of the pizza did the girls eat together?
>
> Solution: The amount of pizza Mary ate can be thought of a $\frac{3}{6}$ or $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$. The amount of pizza Lacey ate can be thought of a $\frac{1}{6}+\frac{1}{6}$. The total amount of pizza they ate is $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$ or $\frac{5}{6}$ of the whole pizza.

A separate algorithm for mixed numbers in addition and subtraction is not necessary. Students will tend to add or subtract the whole numbers first and then work with the fractions using the same strategies they have applied to problems that contained only fractions.

Example:

Susan and Avery need $8\frac{3}{8}$ feet of ribbon to package gift baskets. Susan has $3\frac{1}{8}$ feet of ribbon and Avery has $5\frac{3}{8}$ feet of ribbon. How much ribbon do they have altogether? Will it be enough to complete the project? Explain why or why not.

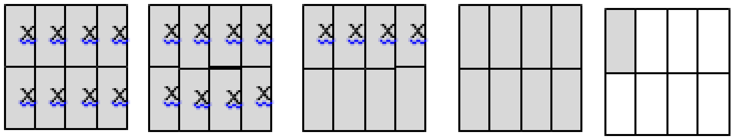
The student thinks:
I can add the ribbon Susan has to the ribbon Avery has to find out how much ribbon they have altogether. Susan has $3\frac{1}{8}$ feet of ribbon and Avery has $5\frac{3}{8}$ feet of ribbon so I can write this as $3\frac{1}{8} + 5\frac{3}{8}$. I know they have 8 feet of ribbon by adding the 3 and 5. They also have $\frac{1}{8}$ and $\frac{3}{8}$ which makes a total of $\frac{4}{8}$ more. Altogether they have $8\frac{4}{8}$ feet of ribbon. $8\frac{4}{8}$ is larger than $8\frac{3}{8}$ so they will have enough ribbon to complete the project. They will even have a little extra ribbon left, $\frac{1}{8}$ foot.



Example:

Timothy has $4\frac{1}{8}$ pizzas left over from his soccer party. After giving some pizza to his friend, he has $2\frac{1}{8}$ of a pizza left. How much pizza did Timothy give to his friend?

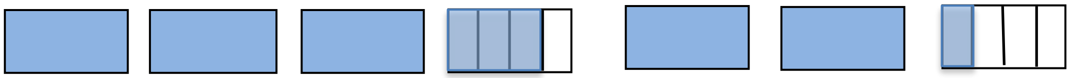
Solution: Timothy had $4\frac{1}{8}$ pizzas to start. This is $\frac{33}{8}$ of a pizza. The x's show the pizza he has left which is $2\frac{4}{8}$ pizzas or $\frac{20}{8}$ pizzas. The shaded rectangles without the x's are the pizza he gave to his friend which is $\frac{13}{8}$ or $1\frac{5}{8}$ pizzas.



Mixed numbers are introduced for the first time in Fourth Grade. Students should have ample experiences of adding and subtracting mixed numbers where they work with mixed numbers and improper fractions. Keep in mind **Concrete-Representation-Abstract (CRA)** approach to teaching fractions. Students need to be able to "show" their thinking using concrete and/or representations BEFORE they move to abstract thinking.

Example:

While solving the problem $3\frac{3}{4}+2\frac{1}{4}$ students could do the following:



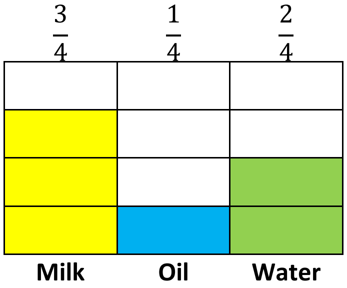
Student 1	Student 2	Student 3
$3+2=5\ and\ \frac{3}{4}+\frac{1}{4}=1$ so 5 + 1 = 6	$3\frac{3}{4}+2=5\frac{3}{4}$ $5\frac{3}{4}+\frac{1}{4}=5\frac{4}{4}\ or\ 6$	$3\frac{3}{4}=\frac{15}{4}\ and\ 2\frac{1}{4}=\frac{9}{4}\ so\ \frac{15}{4}+\frac{9}{4}=\frac{24}{4}or\ 6$

Example:

A cake recipe calls for you to use $\frac{3}{4}$ cup of milk, $\frac{1}{4}$ cup of oil, and $\frac{2}{4}$ cup of water. How much liquid was needed to make the cake? Use an area model to solve.

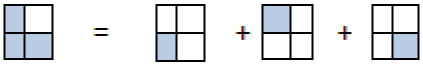
One example area model solution:

$\frac{6}{4}\ or\ 1$ whole cup and $\frac{1}{2}$ more



Instructional Strategies:

In Grade 3, students added unit fractions with the same denominator. Now, students will begin to represent a fraction by decomposing the fraction as the sum of unit fraction and justify with a fraction model. For example, $\frac{3}{4}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$



Students also represented whole numbers as fractions. They use this knowledge to add and subtract mixed numbers with like denominators using properties of number and appropriate fraction models. It is important to stress that whichever model is used, it should be the same for the same whole. For example, a circular model and a rectangular model should not be used in the same problem.

Tools/Resources:

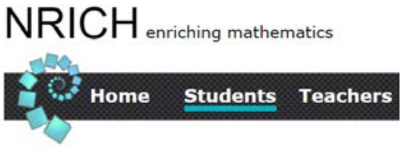
Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.B
  - Comparing two different pizzas
- 4.NF.B.3a
  - Comparing Sums of Unit Fractions
- 4.NF.B.3.b
  - Making 22 Seventeenths in Different Ways
- 4.NF.B.3.c
  - Writing a Mixed Number as an Equivalent Fraction
  - Peaches
  - Plastic Building Blocks
  - Cynthia's Perfect Punch
- 4.NF.B.4.c
  - Sugar in six cans of soda

See: “Harry’s Hike”, NCSM, Great Tasks for Mathematics K-5, (2013). Students determine if a given estimate is justifiable and work thorough different forms of modeling to prove or disprove their original hypothesis.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.3 to access resources specifically for this standard.

NRICH Mathematics:



- Keep It Simple
- Egyptian Fractions
- Fibonacci’s Greedy Algorithm

For Additional Information See Progressions for the Common Core State Standards in Mathematics: 3-5 Number and Operations-Fractions

Common Misconceptions: (4.NF.3 through 4.NF.4)

Students think that it does not matter which model to use when finding the sum or difference of fractions within the same problem. They may represent one fraction with a rectangle and the other fraction with a circle. They need to know that the model for all fractions within the same problem need to represent the same whole.

Some students will not see the connection between decomposing whole numbers and decomposing fractions. Make sure you allow students opportunities to see this structure in fractions and connect it to previous experiences.

Domain: Number and Operations - Fractions (NF)

▶ ***Cluster B**: Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.*
(Limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, 100.)

Standard: 4.NF.4

Apply and extend previous understandings of multiplication (refer to 2.OA.3, 2.OA.4, 3.OA.1, 3.NF.1, 3.NF.2) to multiply a fraction by a whole number.

4.NF.4a. Understand a fraction $\frac{a}{b}$ as a multiple of $\frac{1}{b}$. *For example, use a visual fraction model to represent* $\frac{5}{4}$ *as 5 copies of* $\frac{1}{4}$, *recording the conclusion by the equation* $\frac{5}{4} = 5 \cdot \frac{1}{4}$. **(4.NF.4a)**

4.NF.4b. Understand a multiple of $\frac{a}{b}$ as a multiple of $\frac{1}{b}$, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express* $3 \cdot \frac{2}{5}$ *as* $6 \cdot \frac{1}{5}$, *recognizing this product as* $\frac{6}{5}$. *(In general,* $n \cdot \frac{a}{b} = \frac{n \cdot a}{b}$.*).* **(4.NF.4b)**

4.NF.4c. Solve word problems involving multiplication of a fraction by a whole number, (See Table 2) *(e.g. by using visual fraction models and equations to represent the problem.) For example, if each person at a party will eat* $\frac{3}{8}$ *of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?* **(4.NF.4c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 4.NF.3

Explanation and Examples:

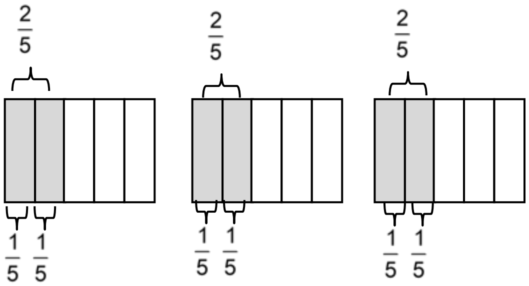
Students need many opportunities to work with problems in context to understand the connections between models and corresponding equations. Contexts involving a whole number times a fraction lend themselves to modeling and examining patterns. This standard builds on students' work of adding fractions and extending that work into multiplication. Allow students to use fraction models and drawings to show their understanding.

When instructing students in multiplying a whole number by a fraction, the **most important** idea is to understand that the whole number describes how many copies are needed of the fraction. For example, I need to figure out how much lasagna I have after a party. There are 5 pans left and each one is $\frac{3}{4}$ full. How many pans of lasagna do I have? The students need to understand that there are 5 copies of $\frac{3}{4}\left(\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right)$. Which means the total in fraction form is $\frac{15}{4}$. Level 3 students should be able to do this type of problem.

Another expectation in 4th grade is that most students should be able to understand the equivalent mixed number form of a fraction greater than one. So most students should be able to understand that $\frac{15}{4}$is equivalent to $3\frac{3}{4}$. When putting the problem in the previous paragraph together with the understanding of the equivalent form, this would be at a DOK level of 4 as it is multiple steps and multiple standards.

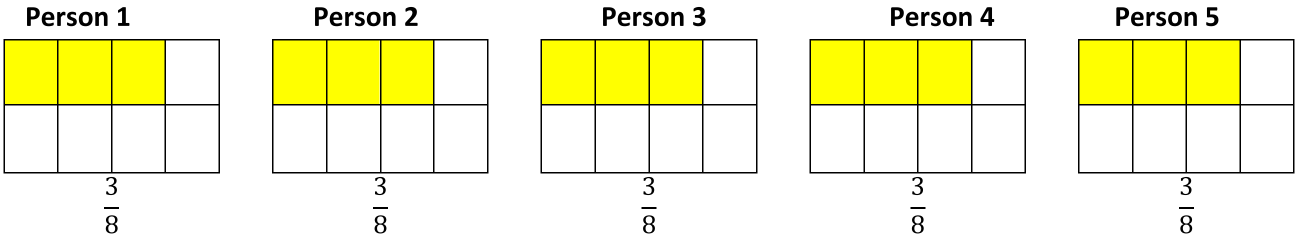
Examples:

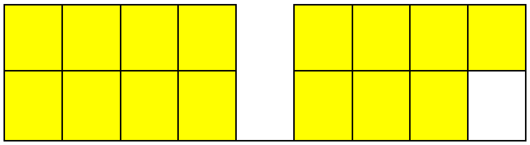
$$3 \times \frac{2}{5} = 6 \times \frac{1}{5} = \frac{6}{5}$$



The example problem in the standards - *If each person at a party eats $\frac{3}{8}$ of a pound of roast beef, and there are 5 people at the party, how many pounds of roast beef are needed? Between what two whole numbers does your answer lie?*

A student may build a fraction model to represent this problem:





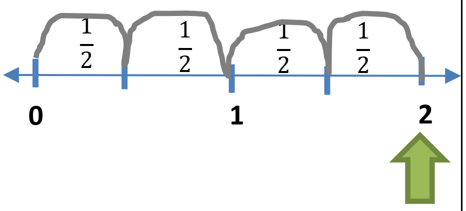
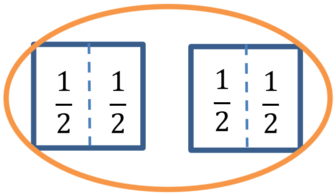
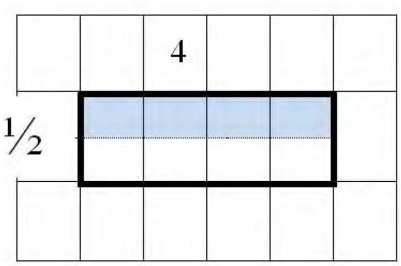
$$\frac{3}{8}+\frac{3}{8}+\frac{3}{8}+\frac{3}{8}+\frac{3}{8}=\frac{15}{8}=1\frac{7}{8}$$

This standard extends the idea of multiplication as repeated addition (4.NF.4b)

For example, $3 \times \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5}\ so\ \frac{6}{5} = 6 \times \frac{1}{5}$. Students are expected to use and create visual fraction models to multiply a whole number by a fraction.

This standard expects students to use **visual fraction models** (**area/region, linear/measurement** and **set models**) to solve word problems related to multiplying a whole number by a fraction. (4.NF.4c)

Example: 4 x $\frac{1}{2}$ = ?

Student 1	**Student 2**	**Student 3**
Draws a number line to show 4 jumps of $\frac{1}{2}$.  	Draws ~~and~~ an area model showing 4 pieces of $\frac{1}{2}$ joined together to equal 2.  	Draws an area model representing $4 \times \frac{1}{2}$ on a grid, dividing each row into $\frac{1}{2}$ to represent the multiplier. 

Instructional Strategies: See 4.NF.3

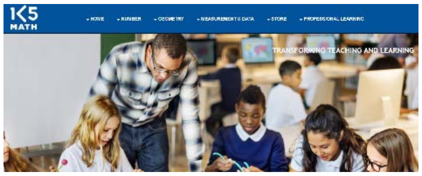
Tools/Resources:

Fraction Tiles, Fraction bars, and patty paper (Area Models); Rulers, Number Lines, and Cuisenaire rods (Linear Models)

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.B.4
    - Extending multiplication from whole numbers to fractions
- 4.NF.B.4.c
    - Sugar in six cans of soda

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.4 to access resources specifically for this standard.



For Additional Information See Progressions for the Common Core State Standards in Mathematics: 3-5 Number and Operations-Fractions

Common Misconceptions: See 4.NF.3

Students often do not use the correct word for the multiplication symbol to create a visual in their mind. When students say “5 times $\frac{1}{2}$”, a visual or picture does not form in their minds, but when they say “5 groups of $\frac{1}{2}$”, a clear picture of five halves of something will form in their minds so the answer of $\frac{5}{2}$ or $2\frac{1}{2}$ is evident.

Domain: Number and Operations – Fractions (NF)

▶ ***Cluster C:*** *Understand decimal notation for fractions, and compare decimal fractions.*
(Students are expected to learn to add decimals by converting them to fractions with the same denominator, in preparation for general fraction addition in grade 5.)

Standard: 4.NF.5

Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. *For example, express* $\frac{3}{10}$ *as* $\frac{30}{100}$, *and add* $\frac{3}{10}+\frac{4}{100}=\frac{34}{100}$. **(4.NF.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: 4.NF.5 through 4.NF.7

This cluster is connected to:

- Developing an understanding of addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers. (4.NF.B)
- Connect with understanding and generating equivalent fractions (4.NF.A).
- Students will perform operations with decimals to hundredths in Grade 5 (5.NBT.B).

Explanation and Examples:

Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators is not a requirement at this grade.

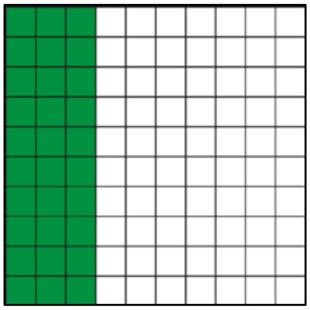
This standard continues the work of equivalent fractions by having students change fractions with denominators of 10 into equivalent fractions denominators of 100. In order for students to have a concrete foundation for future work with decimals (4.NF.6 and 4.NF.7), planning experiences that allow students to shade decimal grids (10x10 grids) can support their understanding.

Students' experiences should focus on working with grids rather than algorithms. Students can use decimal squares, base ten blocks, and other place value models to explore the relationship between fractions with denominators of 10 and denominators of 100, but the main idea is for students to gain a conceptual understanding of decimals. This work in fourth grade lays the foundation for performing operations with decimal numbers in fifth grade.

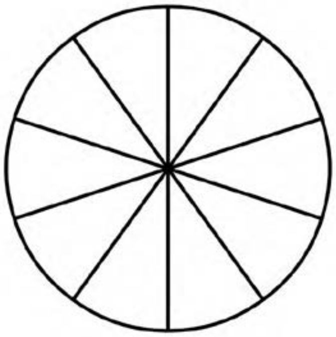
Using pictorial models for decimal work -

Example:

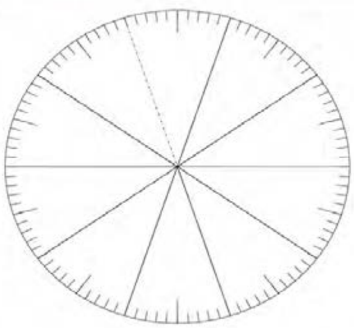
Represent **3 tenths** and **30 hundredths** on the models shown below:
First using Decimal Squares



Or using circle models



10ths circle



100ths circles

Students can use decimal squares, base ten blocks, graph paper, and other place value models to explore the relationship between fractions with denominators of 10 and denominators of 100 (decimal fractions).

Base Ten Blocks: students may represent $\frac{3}{10}$ with 3 of the tens (longs or rods) and may also write the fraction as $\frac{30}{100}$ with the whole in this case being the hundred (the flat would now represent one hundred units with each unit equal to one hundredth). Students begin to make connections to the place value chart as shown in 4.NF.6.

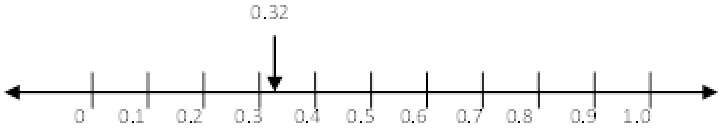
Base Ten blocks have been used for several years as whole number models so changing to a decimal fraction model may not be an easy transfer for some students. In this case, using Decimal Squares as the model for decimals makes better sense for those students. Access this link (https://decimalsquares.com/) for more information about this tool.

Students make connections between fractions with denominators of 10 and 100 and the place value chart. By reading fraction names, students say $\frac{32}{100}$ as thirty-two hundredths and rewrite this as 0.32 or represent it on a place value model as shown below.

Hundreds	Tens	Ones	Tenths	Hundredths
			3	2

Students use the representations explored in 4.NF.5 to understand $\frac{32}{100}$ can be expanded to $\frac{3}{10}$ and $\frac{2}{100}$.

Students represent values such as 0.32 or $\frac{32}{100}$ on a number line. $\frac{32}{100}$ is more than $\frac{30}{100}$ (or $\frac{3}{10}$) and less than $\frac{40}{100}$ (or $\frac{4}{10}$). It is closer to $\frac{30}{100}$ so it would be placed on the number line near that value.



Instructional Strategies:

Students extend fraction equivalence from Grade 3 to include fractions with a denominator of 10. Provide fraction models of tenths and hundredths so that students can express a fraction with a denominator of 10 as an equivalent fraction with a denominator of 100 using the understanding gained from fraction work in third grade.

The place value system developed for whole numbers extends to fractional parts represented as decimals. This makes a wonderful connection and application to the metric system.

Decimals are another way to write fractions. The place-value system developed for whole numbers extends to decimals. The concept of one whole used in fractions is extended to models of decimals.

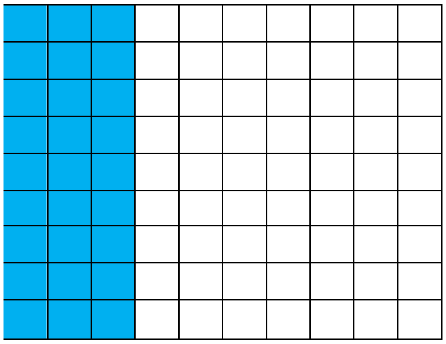
It is important that students make connections between fractions and decimals. They should be able to write decimals for fractions with denominators of 10 or 100. Have students say the fraction with denominators of 10 and 100 aloud. For example $\frac{4}{10}$ would be “four tenths” or $\frac{27}{100}$ would be “twenty-seven hundredths.” Also, have students represent decimals in word form and the decimal place value form.

Students should be able to express decimals to the hundredths as the sum of two or more decimals or fractions. This is based on an understanding of decimal place value and being able to decompose. For example 0.32 would be the sum of 3 tenths and 2 hundredths. Using this understanding students can write 0.32 as the sum of two fractions ($\frac{3}{10} + \frac{2}{100}$)

Students’ understanding of decimals to hundredths is important in preparation for performing operations with decimals to hundredths in Grade 5.

In decimal numbers students will see the continuation of their work with the whole number base ten system. They will see that the value of each place is 10 times the value of the place to its immediate right. Understanding the decimal place value system is important **prior** to a “rule” of “moving the decimal point” when performing operations involving decimals. With no true understanding, students will misapply this “rule” with no understanding of base ten.

When comparing two decimals, remind students that as in comparing two fractions, the decimals need to refer to the same whole. Students should be using visual models frequently in order to compare two decimals. They should be able to explain how the decimal model represents the decimal number and then explain the component parts. They can shade in a representation of each decimal on a 10 x 10 grid and explain why the visual is representing the decimal. In the example below the 10 x 10 grid shown is defined as one whole. The decimal models must relate to the whole, which is a 100 grid in the example.



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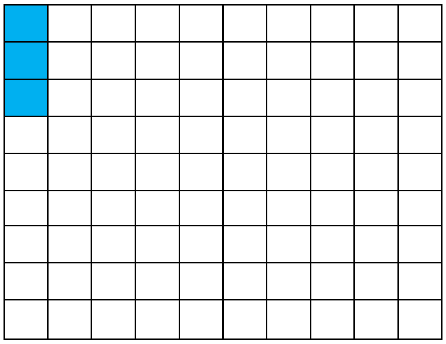


0.03

Flexibility with converting fractions to decimals and decimals to fractions provides efficiency in solving problems involving all four operations in later grades.

Resources/Tools:

- Length or area models
- 10 x 10 square on a grid
- Decimal place-value mats
- Base-ten blocks
- Number lines

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.C.5
  - Expanded Fractions and Decimals
  - Dimes and Pennies
  - Fraction Equivalence
  - How Many Tenths and Hundredths?
  - Adding Tenths and Hundredths

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- "A Meter of Candy" – This series of three hands-on activities for students to develop and reinforce their understanding of hundredths as fractions, decimals and percentages. Students explore using candy pieces and make physically models to connect a set and linear model (meter); produce area models (grids and pie graphs).

Georgia Department of Education:

- "Flag Fractions" - Students create a flag by coloring fractional pieces of the flag and name and write the fractional parts created on their flag. While exploring, students add decimal fractions with like denominators, write decimal fractions as decimals, order two digit decimals, and add two digit decimals.

Decimal Squares website – shows examples and has games students can play.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.5 to access resources specifically for this standard.

Common Misconceptions: 4.NF.5

Students do not have a good visual representation in their mind for tenths and hundredths, so they constantly confuse decimals such as 6 tenths and 6 hundredths. Using models and having them explain their reasoning will help.

Domain: Number and Operations – Fractions (NF)

▶ ***Cluster C:*** *Understand decimal notation for fractions, and compare decimal fractions.*
(Students are expected to learn to add decimals by converting them to fractions with the same denominator, in preparation for general fraction addition in grade 5.)

Standard: 4.NF.6

Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as* $\frac{62}{100}$*; describe a length as 0.62 meters; locate 0.62 on a number line diagram.* **(4.NF.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.NF.5

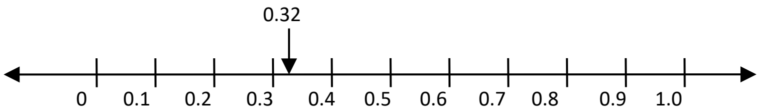
Explanation and Examples:

Students make connections between fractions with denominators of 10 and 100 and the place value chart. By reading fraction names, students say $\frac{25}{100}$ as twenty-five hundredths and rewrite this as 0.25 or represent it on a place value model as shown below.

Hundreds	Tens	Ones	• Tenths	Hundredths
			• 2	5

Students use the representations explored in 4.NF.5 to understand $\frac{25}{100}$ can be expanded to $\frac{2}{10}$ and $\frac{5}{100}$.

Students represent values such as 0.32 or $\frac{32}{100}$ on a number line. $\frac{32}{100}$ is more than $\frac{30}{100}$ (or $\frac{3}{10}$) and less than $\frac{40}{100}$ (or $\frac{4}{10}$). It is closer to $\frac{30}{100}$ so it would be placed on the number line near that value.



Instructional Strategies: See 4.NF.5

Tools/Resources:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.C.5
    - Expanded Fractions and Decimals
    - Dimes and Pennies
    - How Many Tenths and Hundredths?

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.6 to access resources specifically for this standard.

Common Misconceptions: See 4.NF.5

Students do not have a good visual representation in their mind for tenths and hundredths, so they constantly confuse decimals such as 6 tenths and 6 hundredths. Using models and having them explain their reasoning will help.

Domain: Number and Operations – Fractions (NF)

▶ ***Cluster C:*** *<u>Understand decimal notation for fractions, and compare decimal fractions.</u>*
(Students are expected to learn to add decimals by converting them to fractions with the same denominator, in preparation for general fraction addition in grade 5.)

Standard: 4.NF.7

Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the relational symbols >, < =, or ≠, and justify the conclusions, *(e.g. by using a visual model.).* **(4.NF.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

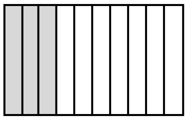
Connections: See 4.NF.5

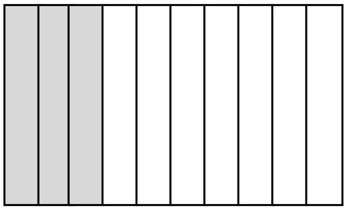
Explanation and Examples:

Mathematically proficient students communicate precisely by engaging in discussion about their reasoning using appropriate mathematical language. The terms students should learn to use with this cluster are**: fraction, numerator, denominator, equivalent, decimals, tenths, hundreds, comparisons/compare, ‹, ›, =,** and ≠.

Students build area models (such as decimal squares) and other models to compare decimals. Through these experiences with fraction models, students will build the understanding that comparisons between decimals or fractions are only valid when the whole is the same for both cases.

Each of the models below shows $\frac{3}{10}$ but the whole on the right is much bigger than the whole on the left. They are both $\frac{3}{10}$ but the model on the right is a much larger whole than the model on the left. I can compare these models visually. In this case would $\frac{3}{10}$ of the first model be equal to $\frac{3}{10}$ of the second model? But when the equation $\frac{3}{10} = \frac{3}{10}$ is written without an indication that the wholes are different then it is always known that the wholes are the same.

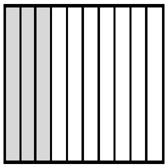


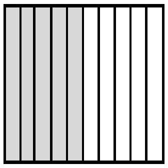


When the wholes are the same, the decimals or fractions can be compared.

Example:

Show or draw a model to demonstrate that 0.3 < 0.5. (Students would show decimal squares of three-tenths and five-tenths OR sketch two models of approximately the same size to show that the area which represents three-tenths is smaller than the area that represents five-tenths.)





Instructional Strategies: See 4.NF.5

Tool/Resources:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.NF.C.7
  - Using Place Value

Georgia Department of Education:

- "Flag Fractions" - Students create a flag by coloring fractional pieces of the flag and name and write the fractional parts created on their flag. While exploring, students add decimal fractions with like denominators, write decimal fractions as decimals, order two digit decimals, and add two digit decimals.

Visit K-5 Math Teaching Resources click on **Number**, then on **4th Grade**. Scroll down to 4.NF.7 to access resources specifically for this standard.

Common Misconceptions:

Students treat decimals as whole numbers when making comparison of two decimals. They think the longer the number, the greater the value. For example, they think that .03 is greater than 0.3.

Domain: Measurement and Data (MD)

◆ ***Cluster A:*** *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

Standard: 4.MD.1

Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs* (*1, 12*), (*2, 24*), (*3, 36*), … **(4.MD.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: 4.MD.1 through 4.MD.3

This cluster is connected to:

- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures (Grade 3.MD.8).
- Geometric measurement; understand concepts of area and relate area to multiplication and to addition. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers (Grade 4.NF.B).

Explanation and Examples:

Mathematically proficient students communicate precisely by engaging in discussion about their reasoning using appropriate mathematical language. The terms students should learn to use with increasing precision in this cluster are: **measure, metric, convert/conversion, relative size, liquid volume, mass, length, distance, kilometer (km), meter (m), centimeter (cm), kilogram (kg), gram (g), liter (L), milliliter (mL), inch (in), foot (ft), yard (yd), mile (mi), ounce (oz), pound (lb), cup (c), pint (pt), quart (qt), gallon (gal), time, hour, minute, second, equivalent, area, perimeter.**

The units of measure that have not been addressed before this grade level are pounds, ounces, kilometers, milliliters, and seconds. Students' prior experiences were limited to measuring length, mass in grams and kilograms, liquid volume in liters, and elapsed time. Students did not convert measurements in prior grades. Students need ample opportunities to become familiar with these new units of measure and to understand conversions with these units of measure.

It will be helpful to use a two-column chart as students are converting from larger to smaller units and recording equivalent measurements. Students can make statements such as; *If one foot is 12 inches, then 3 feet has to be 36 inches because there are 3 groups of 12.*

Example:

kg	g
1	1000
2	2000
3	3000

ft	in
1	12
2	24
3	36

lb	oz
1	16
2	32
3	48

Foundational understandings to help with measure concepts:

1. Understand that larger units can be subdivided into equivalent smaller units (**partition**).
2. Understand that the same unit can be repeated to determine the measure of an object (**iteration**).
3. Understand there is a relationship between the size of a unit and the number of units needed (**compensatory principal**).

Instructional Strategies:

In order for students to have a better understanding of the **relationships** between units, they need to use measuring devices in class to develop a sense of the attributes being measured. The number of units should to relate to the size of the unit. They need to develop an understanding that there are 12 inches in 1 foot and 3 feet in 1 yard.

Allow students to use rulers or a yardstick to discover these **relationships** among units of measurements. Using 12-inch rulers and yardsticks, students will see that three of the 12-inch rulers is the same length as a yardstick, so 3 feet is equivalent to one yard. A similar strategy can be used with rulers marked with centimeters and a meter stick to discover the relationships between centimeters and meters. Have students record measurement relationships in a two column table or t-chart.

Resources/Tools

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.A.1
  - Who is the Tallest?

For detailed information see the **Learning Progression for Measurement.**

- Yardsticks, meter sticks, and rulers (marked with customary and metric units)
- Teaspoons and tablespoons
- Graduated measuring cups (marked with customary and metric units)

Georgia Department of Education:

- "Kilogram Scavenger Hunt" - Students will collect items in the classroom that they think weigh about a kilogram. They will actually weigh the items to see how close their estimates were to the exact weight.

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **4th Grade**. Scroll down to 4.MD.1 to access resources specifically for this standard.

Common Misconceptions: 4.MD.1 through 4.MD.3

Students believe that larger units will give the larger measure.

Students should be given multiple opportunities to measure the same object with different measuring units. For example, have the students measure the length of a room with one-inch tiles, with one-foot rulers, and with yardsticks. Students should notice that it takes fewer yard sticks to measure the room than rulers or tiles and explain their reasoning.

Domain: Measurement and Data (MD)

◆ ***Cluster A:*** *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

Standard: 4.MD.2

Use the four operations to solve word problems (See Table 1 and Table 2) involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale. **(4.MD.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 4.MD.1

Explanation and Examples:

This standard includes multi-step word problems related to expressing measurements from a larger unit in terms of a smaller unit (e.g., feet to inches, meters to centimeters, dollars to cents).

Students should have ample opportunities to use number line diagrams to solve word problems.

Example:
Debbie and 10 friends are planning for a pizza party. They purchased 3 quarts of milk. If each glass holds 8 oz will everyone get at least one glass of milk?

Possible solution: Debbie plus 10 friends = 11 total people
11 people X 8 ounces (glass of milk) = 88 total ounces
1 quart = 2 pints = 4 cups = 32 ounces
Therefore 1 quart = 2 pints = 4 cups = 32 ounces
2 quarts = 4 pints = 8 cups = 64 ounces
3 quarts = 6 pints = 12 cups = 96 ounces

If Debbie purchased 3 quarts (6 pints) of milk there would be enough for everyone at her party to have at least one glass of milk? If each person drank 1 glass, she would have one 8 oz glass (or 1 cup) of milk left over.

Examples with various operations:

Addition: Mason ran for an hour and 15 minutes on Monday, 25 minutes on Tuesday, and 40 minutes on Wednesday. What was the total number of minutes Mason ran?

Subtraction: A pound of apples costs $1.20. Rachel bought a pound and a half of apples. If she gave the clerk a $5.00 bill, how much change will she get back?

Multiplication: Mario and his 2 brothers are selling lemonade. Mario brought one and a half liters, Javier brought 2 liters, and Ernesto brought 450 milliliters. How many total milliliters of lemonade did the boys have?

Division/fractions: Susan has 2 feet of ribbon. She wants to give her ribbon to her three friends so each friend gets the same amount. How much ribbon will each friend get?
Students may record their solutions using fractions of a foot or inches. (The answer would be $\frac{2}{3}$ *of a foot or 8 inches.)*
Students are able to express the answer in inches because they understand that $\frac{1}{3}$ *of a foot is 4 inches and* $\frac{2}{3}$ *of a foot is 2 groups of* $\frac{1}{3}$.

Number line diagrams that feature a measurement scale can represent measurement quantities. Examples include: ruler, diagram marking off distance along a road with cities at various points, a timetable showing hours throughout the day, or a volume measure on the side of a container.

Example:
At 7:00 a.m. Melisa wakes up to go to school. It takes her 8 minutes to shower, 9 minutes to get dressed and 17 minutes to eat breakfast. How many minutes does she have until the bus comes at 8:00 a.m.? Use the number line to help solve the problem.

Instructional Strategies: See 4.MD.1
Students should solve word problems involving **distances, intervals of time, liquid volumes, masses of objects,** and **money**, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit.

Present problems that involve multiplication of a fraction by a whole number (denominators are 2, 3, 4, 5, 6, 8, 10, 12 and 100). Problems involving addition and subtraction of fractions should have the same denominators. Allow students to use strategies learned with these concepts in order to solve measurement problems.

Resources/Tools:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.A.2
    - Margie Buys Apples

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **4th Grade**. Scroll down to 4.MD.2 to access resources specifically for this standard.

Common Misconceptions: See 4.MD.1

Domain: Measurement and Data (MD)

◆ ***Cluster A:*** *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

Standard: 4.MD.3

Apply the area and perimeter formulas for rectangles in real world and mathematical problems explaining and justifying the appropriate unit of measure. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.* **(4.MD.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

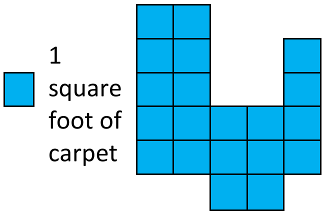
Connections: See 4.MD.1

Explanation and Examples:

Students developed understanding of area and perimeter in 3rd grade by using **visual** models. While students are expected to use formulas to calculate area and perimeter of rectangles, they need to **understand and be able to communicate their understanding of why the formulas work**. This standard expects students to generalize their understanding of **area** and **perimeter** by connecting the concepts to mathematical formulas. These formulas should be developed through experience not just memorization.

The formula for **perimeter** can be $2l + 2w\ or\ 2(l + w)$ and the answer will be in linear units. The formula for **area** is $l \times w$ and the answer will always be in square units (square feet, square meters, square inches, etc.) since multiplication forms a rectangle.

Example:
Mrs. Fields is covering his miniature golf course with artificial grass carpet. How many 1-foot squares of carpet will she need to cover the entire course? The diagram is below:



Instructional Strategies: See 4.MD.1

Students have already used models to find area and perimeter in Grade 3. They now need to relate discoveries from the use of models to forming an understanding of the **area and perimeter formulas** to solve real-world and mathematical problems.

Resources/Tools:

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.A.3
    - Karl's Garden

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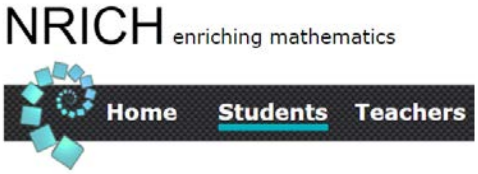
- Areas of Rectangles interactive tool for students

NRICH Mathematics:

- Area and Perimeter activity
- Fence It activity
- Can They Be Equal activity
- Area and Perimeter series of lessons



Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **4th Grade**. Scroll down to 4.MD.3 to access resources specifically for this standard.

Common Misconceptions:

Students frequently confuse area and perimeter. Provide lots of opportunity for students to work with both measures on the same object and have them explain which measure is area and which is perimeter and why? How do they know they are correct?

Domain: Measurement and Data (MD)

◆ ***Cluster B:*** *<u>Represent and interpret data.</u>*

Standard: 4.MD.4

Make a data display (line plot, bar graph, pictograph) to show a set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Solve problems involving addition and subtraction of fractions by using information presented in the data display. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.* **(4.MD.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Understand a fraction as a number on the number line; represent fractions on a number line diagram (3.NF.2).
- Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size (3.NF.3).
- Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters (3.MD.5).
- Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem (4.NF.3d).

Explanation and Examples:

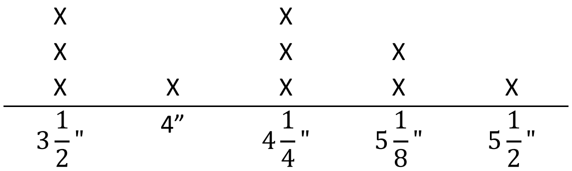
Mathematically proficient students communicate precisely by engaging in discussions about their reasoning using appropriate mathematical language. The terms students should learn to use with increasing precision with this cluster are: **data, data display, line plot, bar graph, pictograph, length, fractions.**

This standard provides a context for students to work with fractions by measuring objects to an eighth of an inch. Students are making data displays of this data, then adding and subtracting fractions based on the data in the data display to answer questions.

Example:

Students measured objects in their desk to the nearest $\frac{1}{2}, \frac{1}{4}$ or $\frac{1}{8}$ inch. They displayed their data collected on a line plot. How many object measured $\frac{1}{4}$ inch? $\frac{1}{2}$ inch? If you put all the objects together end to end what would be the total length of **all** the objects.

Ten students in Room 31 measured their pencils at the end of the day. They recorded their results on the line plot below.



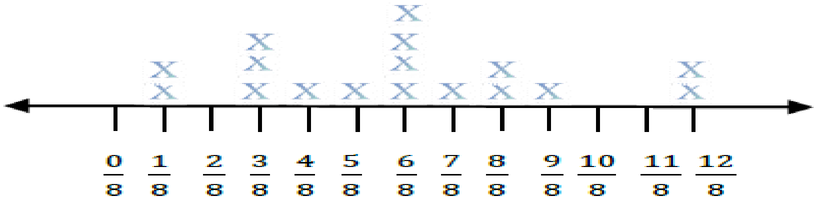
Possible questions:

- What is the difference in length from the longest to the shortest pencil?
- If you were to line up all the pencils, what would the total length be?
- If the $5\frac{1}{8}$" pencils are placed end to end, what would be their total length?

Instructional Strategies:

Data should be measured and then represented on line plots, bar graphs or pictographs in units of whole numbers, halves, quarters, and eighths. Students should then use the data to solve problems involving addition or subtraction of fractions.

Have students create line plots with fractions of a unit ($\frac{1}{2}, \frac{1}{4}$ or $\frac{1}{8}$) and plot data showing multiple data points for each fraction. The line plot below shows measurements of various insects.



Pose questions that students may answer, such as:

- “How many one-eighths are shown on the line plot?” Expect “two one-eighths” as the answer.
- Then ask, “What is the total of these two one-eighths?” Encourage students to count the fractional numbers as they would with whole-number counting, but using the fraction name.
- “What is the total number of inches for insects measuring $\frac{3}{8}$ inches?” Students can use skip counting with fraction names to find the total, such as, “three-eighths, six-eighths, nine-eighths. The last fraction names the total.
- Students should notice that the denominator did not change when they were saying the fraction name.
- Have them make a statement about the result of adding fractions with the same denominator.
- “What is the total number of insects measuring $\frac{1}{8}$ inch and $\frac{5}{8}$ inches? If you connected all those insects, how long would the insect chain be?” Have students write number sentences to represent the problem and solution such as, $\frac{1}{8} + \frac{1}{8} + \frac{5}{8} = \frac{7}{8}$ inches.

Use visual fraction strips and fraction bars to represent problems to solve problems involving addition and subtraction of fractions.

Resources/Tools

- Fraction bars or strips
- Number Lines

For detailed information see Learning Progression for Data.

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.B.4
    - Button Diameters

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- Data Grapher
- Bar Grapher

See EngageNY Module 5, Topic E, F, & G for Lessons.

Visit K-5 Math Teaching Resources click on **Measurement and Data**, then on **4th Grade**. Scroll down to 4.MD.4 to access resources specifically for this standard.

See also : "Bugs, Giraffes, Elephants, and More", NCSM, Great Tasks for Mathematics K-5. (2013). Students interpret line plots with scales written to the nearest quarter of a unit.

Common Misconceptions:

Students use whole-number names when counting fractional parts on a number line. The fraction name should be used instead. For example, if two-fourths is represented on the line plot three times, then there would be six-fourths.

Students also count the tick marks on the number line to determine the fraction, rather than looking at the "distance" or "space" between the marks.

Domain: Geometry (G)

● ***Cluster A:*** *<u>Draw and identify lines and angles, and classify shapes by properties of their lines and angles</u>.*

Standard: 4.G.1

Draw points, lines, line segments, rays, angles (right, acute, obtuse, straight, reflex), and perpendicular and parallel lines. Identify these in two-dimensional figures. **(4.G.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: 4.G.1 through 4.G.3

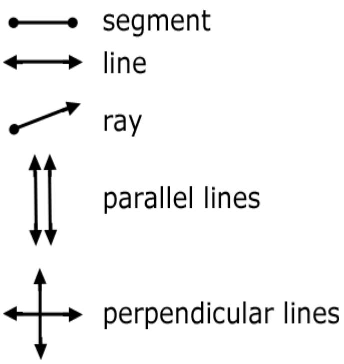
This cluster is connected to:

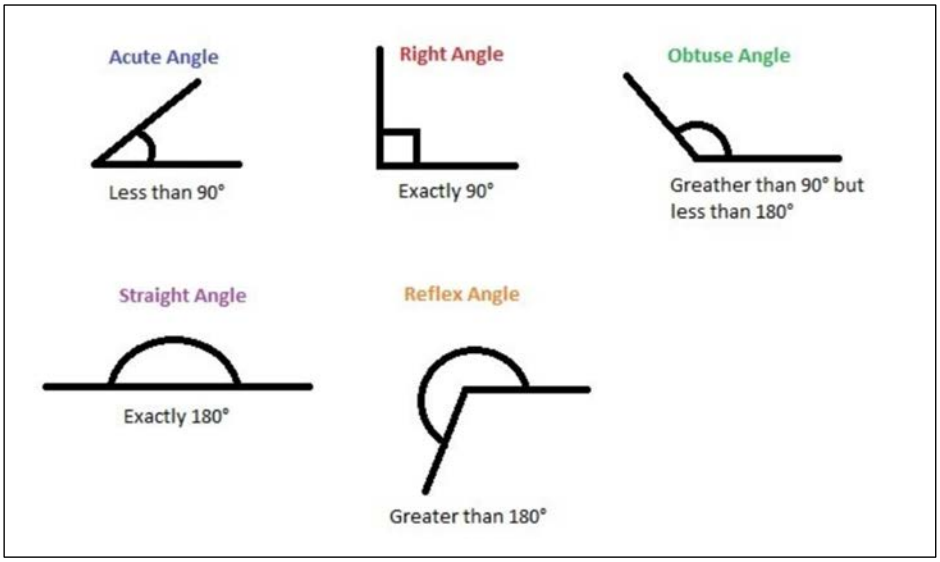
- Understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, particular angle measures, and symmetry.
- Geometric measurement: understand concepts of angles and measure angles (4.MD.3). Symmetry can be related to experiences in art.

Explanation and Examples:

This standard expects students to draw component parts of two-dimensional geometric figures and to identify them within two-dimensional figures. This is the first time that students are exposed to rays, angles, perpendicular lines and parallel lines.

Examples of points, line segments, lines, angles, parallelism, and perpendicularity can be seen daily. Students do not easily identify lines and rays because they are more abstract. These are lines and so they "go on infinitely" in one or two directions. Line segments have end points so they are more easily understood.





Example:

Draw two different types of quadrilaterals that have two pairs of parallel sides?

Is it possible to have an acute right triangle? Justify your reasoning using pictures and words.

Example:

How many acute, obtuse and right angles are in this shape?

Draw and list the properties of a parallelogram. Draw and list the properties of a rectangle. How are your drawings and lists alike? How are they different? Be ready to share your thinking with the class.

Instructional Strategies:

Angles

Students can and should make geometric distinctions about angles without measuring or mentioning degrees. Angles should be classified in comparison to right angles, such as greater than, less than, or the same size as a right angle.

Students can use the corner of a sheet of paper as a benchmark for a right angle. They can use a right angle to determine relationships of other angles.

With the use of a dynamic geometric program, students can easily construct points, lines and geometric figures. They can also draw lines perpendicular or parallel to other line segments.

Resources/Tools

- Mirrors, Miras
- Geoboards
- GeoGebra is a free dynamic software for learning and teaching

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.G.A.1
    - The Geometry of Letters
    - What's the Point?
    - Measuring Angles

Visit K-5 Math Teaching Resources click on **Geometry**, then on **4th Grade**. Scroll down to 4.G.1 to access resources specifically for this standard.

Common Misconceptions:

Students believe a wide angle with short sides may seem smaller than a narrow angle with long sides. Students can compare two angles by tracing one and placing it over the other. Students will then realize that the length of the sides does not determine whether one angle is larger or smaller than another angle. The measure of the angle does not change.

Domain: Geometry (G)

● ***Cluster A:*** *Draw and identify lines and angles, and classify shapes by properties of their lines and angles.*

Standard: 4.G.2

Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles (right, acute, obtuse, straight, reflex). Recognize and categorize triangles based on angles (acute, obtuse, equiangular, and right) and/or sides (scalene, isosceles, and equilateral). **(4.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 4.G.1

Explanation and Examples:

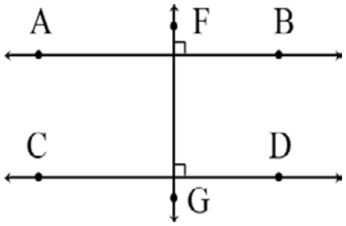
Two-dimensional figures may be classified using different characteristics such as, parallel or perpendicular lines or by angle measurement.

Parallel or Perpendicular Lines:

Students should become familiar with the concept of parallel and perpendicular lines. Two lines are parallel if they never intersect and are always equidistant. Two lines are perpendicular if they intersect in right angles (90°).

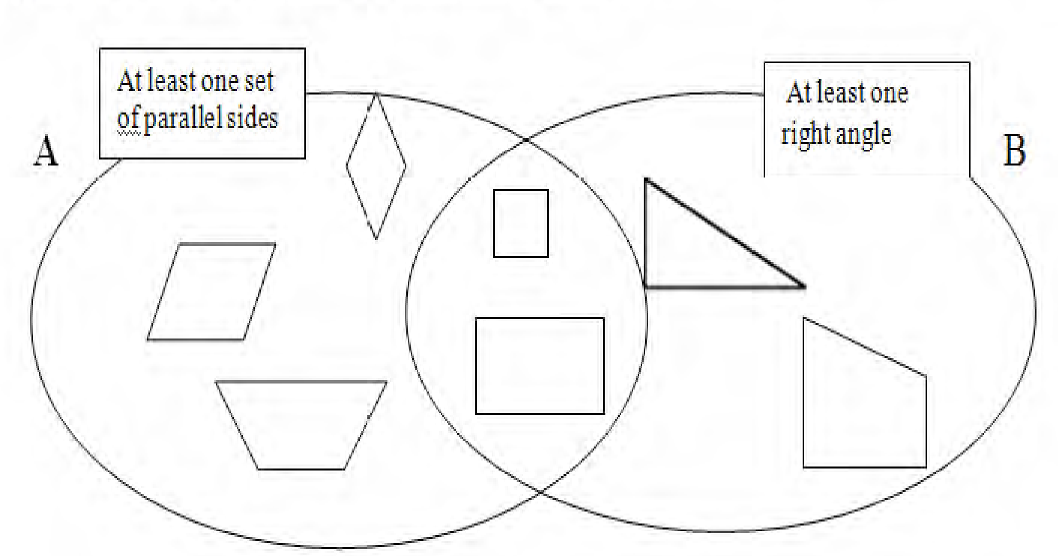
Students may use transparencies with lines to arrange two lines in different ways to determine that the 2 lines might intersect in one point or may never intersect. Further investigations may be initiated using geometry software. These types of explorations may lead to a discussion on angles.

Parallel and perpendicular lines shown below:



This standard calls for students to sort objects based on parallelism, perpendicularity and angle types.

Example:



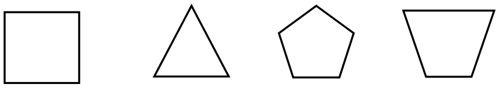
Do you agree with the label on each of the circles in the Venn diagram above? Describe why some shapes fall in the overlapping sections of the circles.

Example:
Draw and name a figure that has two parallel sides and exactly 2 right angles.

Example:
For each of the following, sketch an example if it is possible. If it is impossible, say so, and explain why or show a counter example.

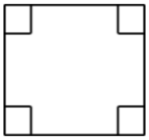
- A parallelogram with exactly one right angle.
- An isosceles right triangle.
- A rectangle that is *not* a parallelogram. *(impossible)*
- Every square is a quadrilateral.
- Every trapezoid is a parallelogram. *(impossible)*

Example:
Identify which of these shapes have perpendicular or parallel sides and justify your selection.



A possible justification that students might give is:

The square has perpendicular lines because the sides meet at a corner, forming right angles.



Categorizing Triangles
Right triangles can be a category for classification. A right triangle has one right angle. There are different types of right triangles. An isosceles right triangle has two or more congruent sides and a scalene right triangle has no congruent sides.

Instructional Strategies: See 4.G.1

Tools/Resources

Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.C.7
    - Finding an unknown angle
- 4.G.A.2
    - Are these right?
    - What shape am I?
    - Defining Attributes of Rectangles and Parallelograms
    - What is a Trapezoid? (Part 1)

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flip books requires membership access, check with your school/district to see if they have an institutional membership which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- Exploring Properties of Triangles and Quadrilaterals

Georgia Department of Education:

- "Quadrilateral Challenge" - Working in pairs, students create the following quadrilaterals. They will identify the attributes of each quadrilateral, then compare and contrast the attributes of different quadrilaterals.

Visit K-5 Math Teaching Resources click on **Geometry**, then on **4th Grade**. Scroll down to 4.G.2 to access resources specifically for this standard.

Common Misconceptions: See 4.G.1

Domain: Geometry (G)

● ***Cluster A:*** *Draw and identify lines and angles, and classify shapes by properties of their lines and angles.*

Standard: 4.G.3

Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry. **(4.G.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 4.G.1

Explanation and Examples:

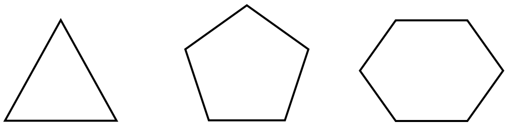
Students need experiences with figures which are symmetrical and non-symmetrical. Figures include both regular and non-regular polygons. Folding cut-out figures will help students determine whether a figure has one or more lines of symmetry.

This standard only includes line symmetry not rotational symmetry.

Example:

For each figure, draw all of the lines of symmetry. What pattern do you notice? How many lines of symmetry do you think there would be for regular polygons with 9 and 11 sides. Sketch each figure and check your predictions.

Polygons with an odd number of sides have lines of symmetry that go from a midpoint of a side through a vertex.



Instructional Strategies:

Give student experience with many shapes that can be folded to determine if they have symmetry. Block letter of the alphabet is one set that students can explore. Students can search magazines to find shapes that are symmetrical and fold to show the line of symmetry.

The use of Miras help students see and draw line to show symmetry. The reflection from the Mira or a mirror helps students see symmetry. Pattern blocks and tangrams are also useful tools in discovering symmetry.

When introducing lines of symmetry, provide examples of geometric shapes with and without lines of symmetry. Shapes can be classified by the existence of lines of symmetry in sorting activities. This can be done informally by folding paper, tracing, creating designs with tiles or investigating reflections in mirrors.

Resources/Tools

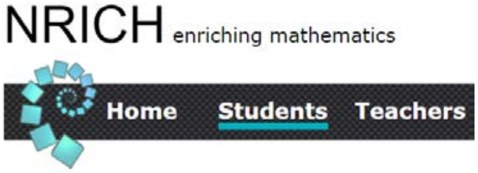
Illustrative Mathematics Grade 4 tasks: **Scroll to the appropriate section to find named tasks.**

- 4.G.A.3
    - Lines of symmetry for triangles
    - Lines of symmetry for quadrilaterals
    - Lines of symmetry for circles
    - Finding Lines of Symmetry

Visit K-5 Math Teaching Resources click on **Geometry**, then on **4th Grade**. Scroll down to 4.G.3 to access resources specifically for this standard.

NRICH Mathematics:

- Reflecting Squarely
- Let Us Reflect
- National Flags



Common Misconceptions:

Some children may think that there can only be one line of symmetry for an object. Encourage them to try folding shapes in more than one way. Giving students multiple copies of the same shapes could help avoid confusion. Coloring one side of the line one color and the other side of the line a different color may aid in seeing multiple lines. In essence the student is seeing if the shape can be folded into $\frac{1}{2}$ halves.

APPENDIX: TABLE 1. Common Addition and Subtraction Situations

Shading taken from OA progression

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
Put Together/ Take Apart[2]	**Total Unknown**	**Addend Unknown**	**Both Addends Unknown[1]**
	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
Compare[3]	**Difference Unknown**	**Bigger Unknown**	**Smaller Unknown**
	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[1] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

[2] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[3] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

TABLE 2. Common Multiplication and Division Situations

Grade level identification of introduction of problem situations taken from OA progression

	Unknown Product	**Group Size Unknown** **("How many in each group?" Division)**	**Number of Groups Unknown** **("How many groups?" Division)**
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	a × b = ?	a × ? = p, and p ÷ a = ?	? × b = p, and p ÷ b = ?

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change

TABLE 3. The Properties of Operations

the subject of the comparing sentence (“A red hat costs n times as much as the blue hat” results in the same comparison as “A blue hat is 1/n times as much as the red hat” but has a different subject.)

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c = a+(b+c)$	$(78+25)+75 = 78+(25+75)$
Commutative	$a+b = b+a$	$2+98 = 98+2$
Additive Identity	$a+0 = a \;\; and \;\; 0+a = a$	$9875+0 = 9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a + -a = -a + a = 0$	$-47+47 = 0$
Properties of Multiplication		
Associative	$(a \times b) \times c = a \times (b \times c)$	$(32 \times 5) \times 2 = 32 \times (5 \times 2)$
Commutative	$a \times b = b \times a$	$10 \times 38 = 38 \times 10$
Multiplicative Identity	$a \times 1 = a \;\; and \;\; 1 \times a = a$	$387 \times 1 = 387$
Multiplicative Inverse	*For every real number a,* $a \neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$	$\frac{8}{3} \times \frac{3}{8} = 1$
Distributive Property of Multiplication over Addition		
Distributive	$a \times (b+c) = a \times b + a \times c$	$7 \times (50+2) = 7 \times 50 + 7 \times 2$

(Variables a, b, and c represent real numbers.)
*Excerpt from NCTM’s **Developing Essential Understanding of Algebraic Thinking**, grades 3-5 p. 16-17*

TABLE 4. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

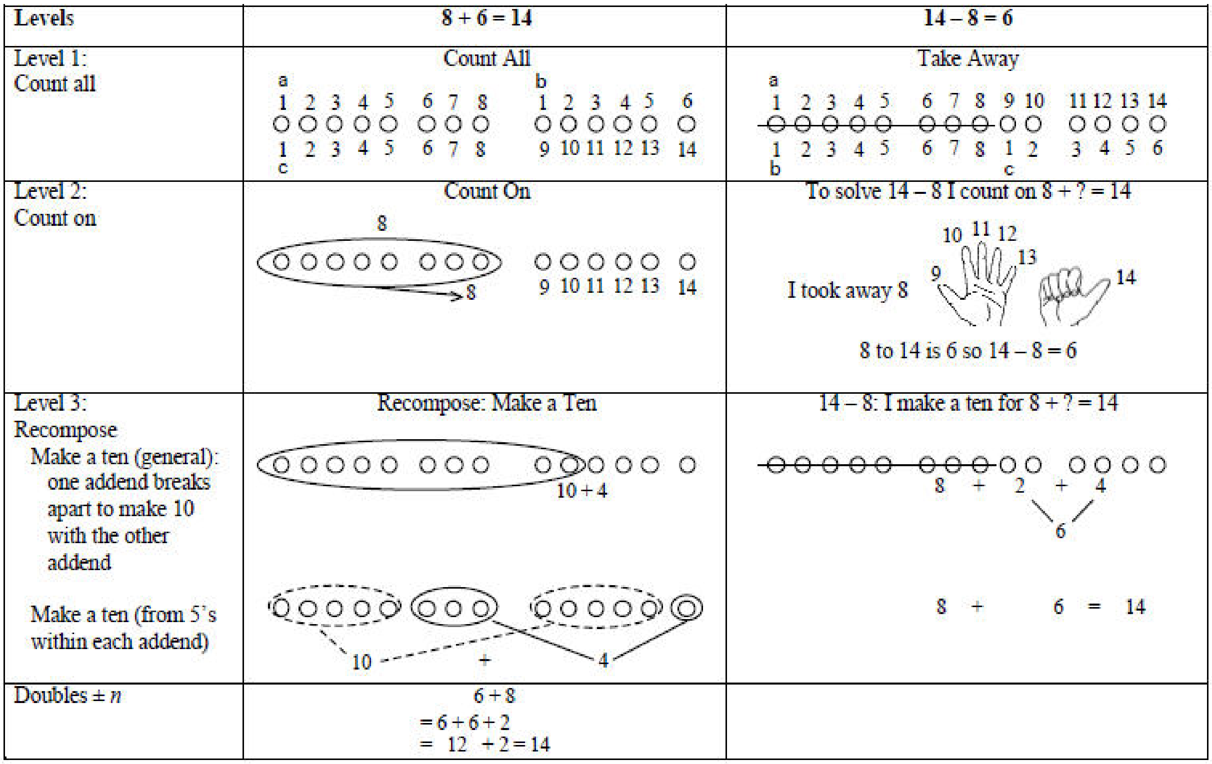
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children



Levels	8 + 6 = 14	14 – 8 = 6
Level 1: Count all	Count All a: 1 2 3 4 5 6 7 8 ○○○○○ ○○○ 1 2 3 4 5 6 7 8 c b: 1 2 3 4 5 6 ○○○○○ ○ 9 10 11 12 13 14	Take Away a: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~○○ ○○○○ 1 2 3 4 5 6 7 8 b 1 2 3 4 5 6 c
Level 2: Count on	Count On 8 ○○○○○ ○○○ ○○○○○ ○ 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8 9 10 11 12 13 14 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten ○○○○○ ○○○ ○○○○○ ○ 10 + 4 ○○○○○ ○○○ ○○○○○ ○ 10 + 4	14 – 8: I make a ten for 8 + ? = 14 ~~○○○○○ ○○○~~○○ ○○○○ 8 + 2 + 4 6 8 + 6 = 14
Doubles ± *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) collection of items and understands that the last word tells "how many" ~~even~~. Beyond on small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words.

Level 3 - At this level the student begins using known facts to solve for unknown facts. For example, the student uses "make a ten" where one addend breaks apart to make 10 with another addend OR a doubles plus/minus one strategy. Students begin to implicitly use the properties of operations.

Table 7: Cognitive Rigor Matrix/Depth of Knowledge (DOK)

The Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	**DOK Level 1** **Recall & Reproduction**	**DOK Level 2** **Basic Skills & Concepts**	**DOK Level 3** **Strategic Thinking & Reasoning**	**DOK Level 4** **Extended Thinking**
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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