

Progressions for the Common Core State Standards in Mathematics (draft)

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6-7, Ratios and Proportional Relationships

Overview

The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades. Ratios and proportional relationships are foundational for further study in mathematics and science and useful in everyday life. Students use ratios in geometry and in algebra when they study similar figures and slopes of lines, and later when they study sine, cosine, tangent, and other trigonometric ratios in high school. Students use ratios when they work with situations involving constant rates of change, and later in calculus when they work with average and instantaneous rates of change of functions. An understanding of ratio is essential in the sciences to make sense of quantities that involve derived attributes such as speed, acceleration, density, surface tension, electric or magnetic field strength, and to understand percentages and ratios used in describing chemical solutions. Ratios and percentages are also useful in many situations in daily life, such as in cooking and in calculating tips, miles per gallon, taxes, and discounts. They also are also involved in a variety of descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g., birth rate, per capita income, body mass index, rain fall, and crop yield) and underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Ratios, rates, proportional relationships, and percent Ratios arise in situations in which two (or more) quantities are related. • Sometimes the quantities have the same units (e.g., 3 cups of apple juice and 2 cups of grape juice), other times they do not (e.g., 3 meters and 2 seconds). Some authors distinguish ratios from rates, using the term "ratio" when units are the same and "rate" when units are different; others use ratio to encompass both kinds of situations. The

- In the Standards, a quantity involves measurement of an attribute. Quantities may be discrete, e.g., 4 apples, or continuous, e.g., 4 inches. They may be measurements of physical attributes such as length, area, volume, weight, or other measurable attributes such as income. Quantities can vary with respect to another quantity. For example, the quantities "distance between the earth and the sun in miles," "distance (in meters) that Sharoya walked," or "my height in feet" vary with time.

Standards use ratio in the second sense, applying it to situations in which units are the same as well as to situations in which units are different. Relationships of two quantities in such situations may be described in terms of ratios, rates, percents, or proportional relationships.

A ratio associates two or more quantities. Ratios can be indicated in words as “3 to 2” and “3 for every 2” and “3 out of every 5” and “3 parts to 2 parts.” This use might include units, e.g., “3 cups of flour for every 2 eggs” or “3 meters in 2 seconds.” Notation for ratios can include the use of a colon, as in 3 : 2. The quotient $\frac{3}{2}$ is sometimes called the value of the ratio 3 : 2.

Ratios have associated rates. For example, the ratio 3 feet for every 2 seconds has the associated rate $\frac{3}{2}$ feet for every 1 second; the ratio 3 cups apple juice for every 2 cups grape juice has the associated rate $\frac{3}{2}$ cups apple juice for every 1 cup grape juice. In Grades 6 and 7, students describe rates in terms such as “for each 1,” “for each,” and “per.” The *unit rate* is the numerical part of the rate; the “unit” in “unit rate” is often used to highlight the 1 in “for each 1” or “for every 1.”

Equivalent ratios arise by multiplying each measurement in a ratio pair by the same positive number. For example, the pairs of numbers of meters and seconds in the margin are in equivalent ratios. Such pairs are also said to be in the same ratio. Proportional relationships involve collections of pairs of measurements in equivalent ratios. In contrast, a proportion is an equation stating that two ratios are equivalent. Equivalent ratios have the same unit rate.

The pairs of meters and seconds in the margin show distance and elapsed time varying together in a *proportional relationship*. This situation can be described as “distance traveled and time elapsed are proportionally related,” or “distance and time are directly proportional,” or simply “distance and time are proportional.” The proportional relationship can be represented with the equation $d = \left(\frac{3}{2}\right)t$. The factor $\frac{3}{2}$ is the constant unit rate associated with the different pairs of measurements in the proportional relationship; it is known as a *constant of proportionality*.

The word *percent* means “per 100” (*cent* is an abbreviation of the Latin *centum* “hundred”). If 35 milliliters out of every 100 milliliters in a juice mixture are orange juice, then the juice mixture is 35% orange juice (by volume). If a juice mixture is viewed as made of 100 equal parts, of which 35 are orange juice, then the juice mixture is 35% orange juice.

More precise definitions of the terms presented here and a framework for organizing and relating the concepts are presented in the Appendix.

Recognizing and describing ratios, rates, and proportional relationships “For each,” “for every,” “per,” and similar terms distinguish situations in which two quantities have a proportional rela-

- In everyday language, the word “ratio” sometimes refers to the value of a ratio, for example in the phrases “take the ratio of price to earnings” or “the ratio of circumference to diameter is π .”

Representing pairs in a proportional relationship

Sharoya walks 3 meters every 2 seconds. Let d be the number of meters Sharoya has walked after t seconds. d and t are in a proportional relationship.

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

d and t are related by the equation $d = \left(\frac{3}{2}\right)t$. Students sometimes use the equals sign incorrectly to indicate proportional relationships, for example, they might write “3 m = 2 sec” to represent the correspondence between 3 meters and 2 seconds. In fact, 3 meters is not equal to 2 seconds. This relationship can be represented in a table or by writing “3 m \rightarrow 2 sec.” Note that the unit rate appears in the pair $\left(\frac{3}{2}, 1\right)$.

relationship from other types of situations. For example, without further information “2 pounds for a dollar” is ambiguous. It may be that pounds and dollars are proportionally related and *every* two pounds costs a dollar. Or it may be that there is a discount on bulk, so weight and cost do not have a proportional relationship. Thus, recognizing ratios, rates, and proportional relationships involves looking for structure (MP7). Describing and interpreting descriptions of ratios, rates, and proportional relationships involves precise use of language (MP6).

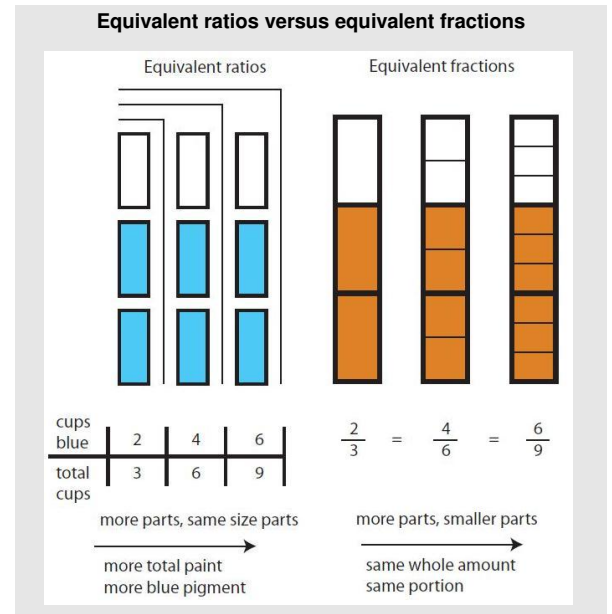
Representing ratios, collections of equivalent ratios, rates, and proportional relationships Because ratios and rates are different and rates will often be written using fraction notation in high school, ratio notation should be distinct from fraction notation.

Together with tables, students can also use tape diagrams and double number line diagrams to represent collections of equivalent ratios. Both types of diagrams visually depict the relative sizes of the quantities.

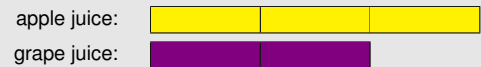
Tape diagrams are best used when the two quantities have the same units. They can be used to solve problems and also to highlight the multiplicative relationship between the quantities.

Double number line diagrams are best used when the quantities have different units (otherwise the two diagrams will use different length units to represent the same amount). Double number line diagrams can help make visible that there are many, even infinitely many, pairs in the same ratio, including those with rational number entries. As in tables, unit rates appear paired with 1.

A collection of equivalent ratios can be graphed in the coordinate plane. The graph represents a proportional relationship. The unit rate appears in the equation and graph as the slope of the line, and in the coordinate pair with first coordinate 1.

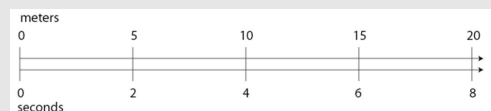


Representing ratios with tape diagrams



This diagram can be interpreted as representing any mixture of apple juice and grape juice with a ratio of 3 to 2. The total amount of juice is represented as partitioned into 5 parts of equal size, represented by 5 rectangles. For example, if the diagram represents 5 cups of juice mixture, then each of these rectangles represents 1 cup. If the total amount of juice mixture is 1 gallon, then each part represents $\frac{1}{5}$ gallon and there are $\frac{3}{5}$ gallon of apple juice and $\frac{2}{5}$ gallon of grape juice.

Representing ratios with double number line diagrams



On double number line diagrams, if A and B are in the same ratio, then A and B are located at the same distance from 0 on their respective lines. Multiplying A and B by a positive number p results in a pair of numbers whose distance from 0 is p times as far. So, for example, 3 times the pair 2 and 5 results in the pair 6 and 15 which is located at 3 times the distance from 0.

Grade 6

Representing and reasoning about ratios and collections of equivalent ratios Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole number measurements such as “3 lemons for every \$1” or “for every 5 cups grape juice, mix in 2 cups peach juice” lend themselves to being recorded in a table.^{6.RP.3a} Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.^{6.RP.1,6.RP.2} It is important for students to focus on the meaning of the terms “for every,” “for each,” “for each 1,” and “per” because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.^{6.EE.9}

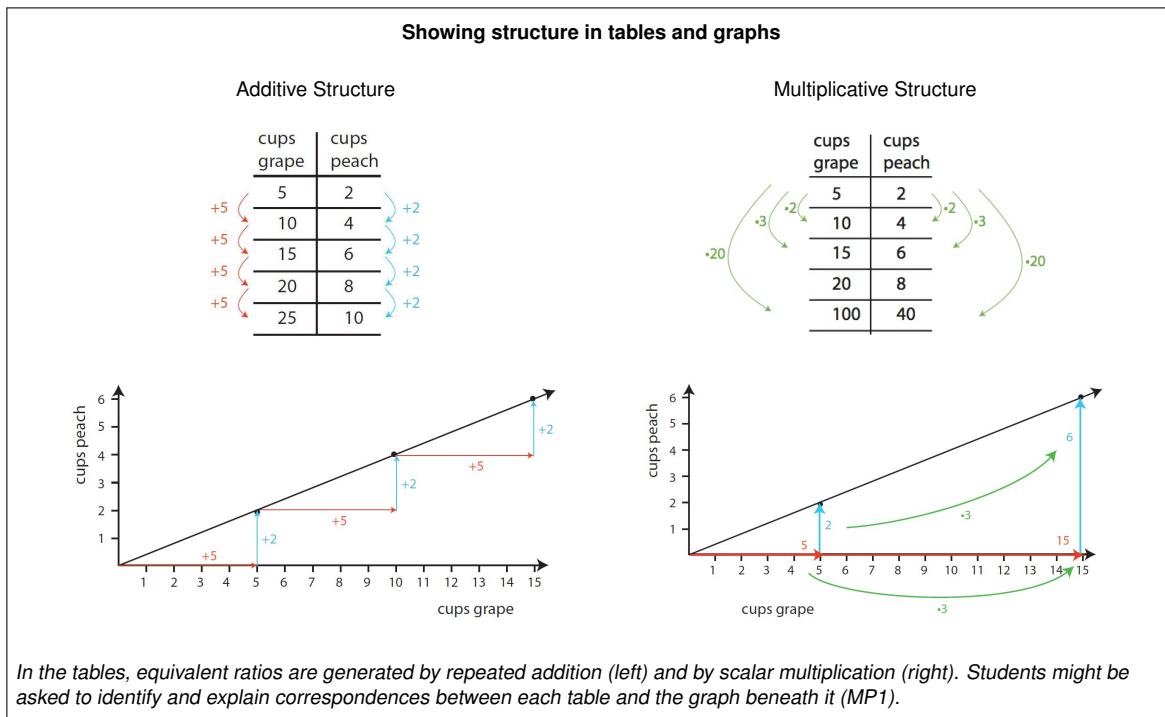
6.RP.3a Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

- a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.



By reasoning about ratio tables to compare ratios,^{6.RP.3a} students can deepen their understanding of what a ratio describes in a context and what quantities in equivalent ratios have in common. For example, suppose Abby's orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint and Zack's orange paint is made by mixing 3 cups red for every 5 cups yellow. Students could discuss that all the mixtures within a single ratio table for one of the paint mixtures are the same shade of orange because they are multiple batches of the same mixture. For example, 2 cups red and 6 cups yellow is two batches of 1 cup red and 3 cups yellow; each batch is the same color, so when the two batches are combined, the shade of orange doesn't change. Therefore, to compare the colors of the two paint mixtures, any entry within a ratio table for one mixture can be compared with any entry from the ratio table for the other mixture.

It is important for students to focus on the rows (or columns) of a ratio table as multiples of each other. If this is not done, a common error when working with ratios is to make additive comparisons. For example, students may think incorrectly that the ratios 1 : 3 and 3 : 5 of red to yellow in Abby's and Zack's paints are equivalent because the difference between the number of cups of red and yellow in both paints is the same, or because Zack's paint could be made from Abby's by adding 2 cups red and 2 cups yellow. The margin shows several ways students could reason correctly to compare the paint mixtures.

Three ways to compare paint mixtures

Same amount of red

Abby's		Zack's	
cups red	cups yellow	cups red	cups yellow
1	3	3	5
2	6	6	10
3	9	9	15
4	12	12	20
5	15	15	25

Same amount of red. Abby's has more yellow, so Abby's is yellower, Zack's is redder.

Same amount of yellow

Abby's		Zack's	
cups red	cups yellow	cups red	cups yellow
1	3	3	5
2	6	6	10
3	9	9	15
4	12	12	20
5	15	15	25

Same amount of yellow. Zack's has more red. So Zack's is redder, Abby's is yellower.

Same total

Abby's			Zack's		
cups red	cups yellow	total cups	cups red	cups yellow	total cups
1	3	4	3	5	8
2	6	8	6	10	16
3	9	12	9	15	24

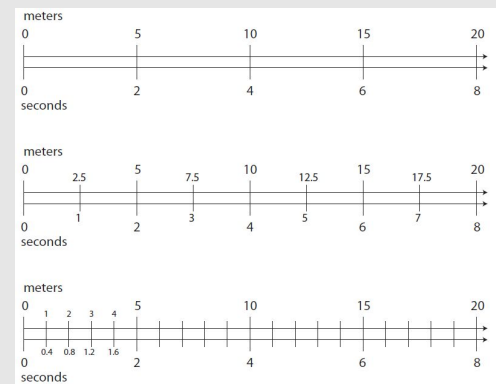
Same total. Abby's has more yellow. Zack's has more red. So Abby's is yellower and Zack's is redder.

Strategies for solving problems Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6. There are a number of strategies for solving problems that involve ratios. As students become familiar with relationships among equivalent ratios, their strategies become increasingly abbreviated and efficient.

For example, suppose grape juice and peach juice are mixed in a ratio of 5 to 2 and we want to know how many cups of grape juice to mix with 12 cups of peach juice so that the mixture will still be in the same ratio. Students could make a ratio table as shown in the margin, and they could use the table to find the grape juice entry that pairs with 12 cups of peach juice in the table. This perspective allows students to begin to reason about proportions by starting with their knowledge about multiplication tables and by building on this knowledge.

As students generate equivalent ratios and record them in tables, their attention should be drawn to the important role of multiplication and division in how entries are related to each other. Initially, students may fill ratio tables with columns or rows of the multiplication table by skip counting, using only whole number entries, and placing these entries in numerical order. Gradually, students should consider entries in ratio tables beyond those they find by skip counting, including larger entries and fraction or decimal entries. Finding

Double number line diagrams used for situations with different units



Double number lines indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.

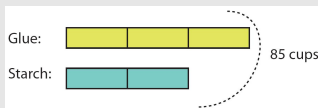
these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N , the distance traveled should also be multiplied (or divided) by N . Double number lines can be useful in representing ratios that involve fractions and decimals.

As students become comfortable with fractional and decimal entries in tables of quantities in equivalent ratios, they should learn to appreciate that unit rates are especially useful for finding entries. A unit rate gives the number of units of one quantity per 1 unit of the other quantity. The amount for N units of the other quantity is then found by multiplying by N . Once students feel comfortable doing so, they may wish to work with abbreviated tables instead of working with long tables that have many values. The most abbreviated tables consist of only two columns or two rows; solving a proportion is a matter of finding one unknown entry in the table.

Measurement conversion provides other opportunities for students to use relationships given by unit rates.^{6.RP.3d} For example, recognizing “12 inches in a foot,” “1000 grams in a kilogram,” or “one kilometer is $\frac{5}{8}$ of a mile” as rates, can help to connect concepts and methods developed for other contexts with measurement conversion.

Representing a problem with a tape diagram

Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?



5 parts \rightarrow 85 cups
 1 part \rightarrow $85 \div 5 = 17$ cups
 3 parts \rightarrow $3 \cdot 17 = 51$ cups
 2 parts \rightarrow $2 \cdot 17 = 34$ cups

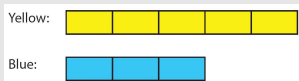
51 cups glue and 34 cups starch are needed.

Tape diagrams can be useful aids for solving problems.

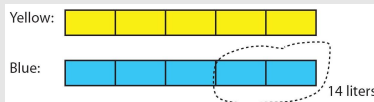
Representing a multi-step problem with two pairs of tape diagrams

Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?

At first:



Then:



2 parts \rightarrow 14 liters
 1 part \rightarrow $14 \div 2 = 7$ liters
 (original total) 8 parts \rightarrow $8 \cdot 7 = 56$ liters

There was 56 liters of green paint to start with.

This problem can be very challenging for sixth or seventh graders.

A progression of strategies for solving a proportion

If 2 pounds of beans cost \$5, how much will 15 pounds of beans cost?

Method 1

pounds	2	4	6	8	10	12	14	1	15
dollars	5	10	15	20	25	30	35	2.50	37.50

“I found 14 pounds costs \$35 and then 1 more pound is another \$2.50, so that makes \$37.50 in all.”

Method 2

pounds	2	1	15
dollars	5	2.50	37.50

“I found 1 pound first because if I know how much it costs for each pound then I can find any number of pounds by multiplying.”

Method 3

pounds	2	15
dollars	5	37.50

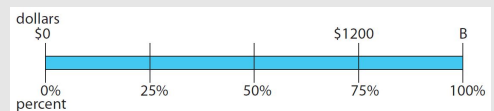
The previous method, done in one step.

With this perspective, the second column is seen as the first column times a number. To solve the proportion one first finds this number.

^{6.RP.3d} Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Solving a percent problem

If 75% of the budget is \$1200, what is the full budget?



“I said 75% is 3 parts and is \$1200
 25% is 1 part and is $1200 \div 3 = \$400$
 100% is 4 parts and is $4 \cdot \$400 = \1600 ”

portion	75	3	1200
whole	100	4	1600

$$75\% \text{ is } \frac{1200}{B}$$

$$\frac{75}{100} = \frac{1200}{B}$$

$$75\% \text{ of } B \text{ is } 1200$$

$$\frac{75}{100} \cdot B = 1200$$

$$B = 1600$$

In reasoning about and solving percent problems, students can use a variety of strategies. Representations such as this, which is a blend between a tape diagram and a double number line diagram, can support sense-making and reasoning about percent.

Grade 7

In Grade 7, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated unit rates. They identify these unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

At this grade, students will also work with ratios specified by rational numbers, such as $\frac{3}{4}$ cups flour for every $\frac{1}{2}$ stick butter.^{7.RP.1} Students continue to use ratio tables, extending this use to finding unit rates.

Recognizing proportional relationships Students examine situations carefully, to determine if they describe a proportional relationship.^{7.RP.2a} For example, if Josh is 10 and Reina is 7, how old will Reina be when Josh is 20? We cannot solve this problem with the proportion $\frac{10}{7} = \frac{20}{R}$ because it is not the case that for every 10 years that Josh ages, Reina ages 7 years. Instead, when Josh has aged 10 another years, Reina will as well, and so she will be 17 when Josh is 20.

For example, if it takes 2 people 5 hours to paint a fence, how long will it take 4 people to paint a fence of the same size (assuming all the people work at the same steady rate)? We cannot solve this problem with the proportion $\frac{2}{5} = \frac{4}{H}$ because it is not the case that for every 2 people, 5 hours of work are needed to paint the fence. When more people work, it will take fewer hours. With twice as many people working, it will take half as long, so it will take only 2.5 hours for 4 people to paint a fence. Students must understand the structure of the problem, which includes looking for and understand the roles of “for every,” “for each,” and “per.”

Students recognize that graphs that are not lines through the origin and tables in which there is not a constant ratio in the entries do not represent proportional relationships. For example, consider circular patios that could be made with a range of diameters. For such patios, the area (and therefore the number of pavers it takes to make the patio) is not proportionally related to the diameter, although the circumference (and therefore the length of stone border it takes to encircle the patio) is proportionally related to the diameter. Note that in the case of the circumference, C , of a circle of diameter D , the constant of proportionality in $C = \pi \cdot D$ is the number π , which is not a rational number.

Equations for proportional relationships As students work with proportional relationships, they write equations of the form $y = cx$, where c is a constant of proportionality, i.e., a unit rate.^{7.RP.2c} They

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

Ratio problem specified by rational numbers: Three approaches

To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you want to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need?

Method 1

cups blue	$\frac{1}{2}$	$\xrightarrow{\cdot 6}$	3	$\xrightarrow{\cdot 4}$	12
cups red	$\frac{1}{3}$		2		8
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	$\xrightarrow{\cdot 6}$	5	$\xrightarrow{\cdot 4}$	20

“I thought about making 6 batches of purple because that is a whole number of cups of purple. To make 6 batches, I need 6 times as much blue and 6 times as much red too. That was 3 cups blue and 2 cups red and that made 5 cups purple. Then 4 times as much of each makes 20 cups purple.”

Method 2

$$\frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10} \quad \frac{6}{10} \cdot 20 = 12$$

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15} \quad \frac{6}{15} \cdot 20 = 8$$

“I found out what fraction of the paint is blue and what fraction is red. Then I found those fractions of 20 to find the number of cups of blue and red in 20 cups.”

Method 3

cups blue	$\frac{1}{2}$	$\xrightarrow{\cdot \frac{3}{5}}$	12
cups red	$\frac{1}{3}$	$\xrightarrow{\cdot \frac{2}{5}}$	8
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	$\xrightarrow{\cdot \frac{2}{5}}$	20

Like Method 2, but in tabular form, and viewed as multiplicative comparisons.

7.RP.2a Recognize and represent proportional relationships between quantities.

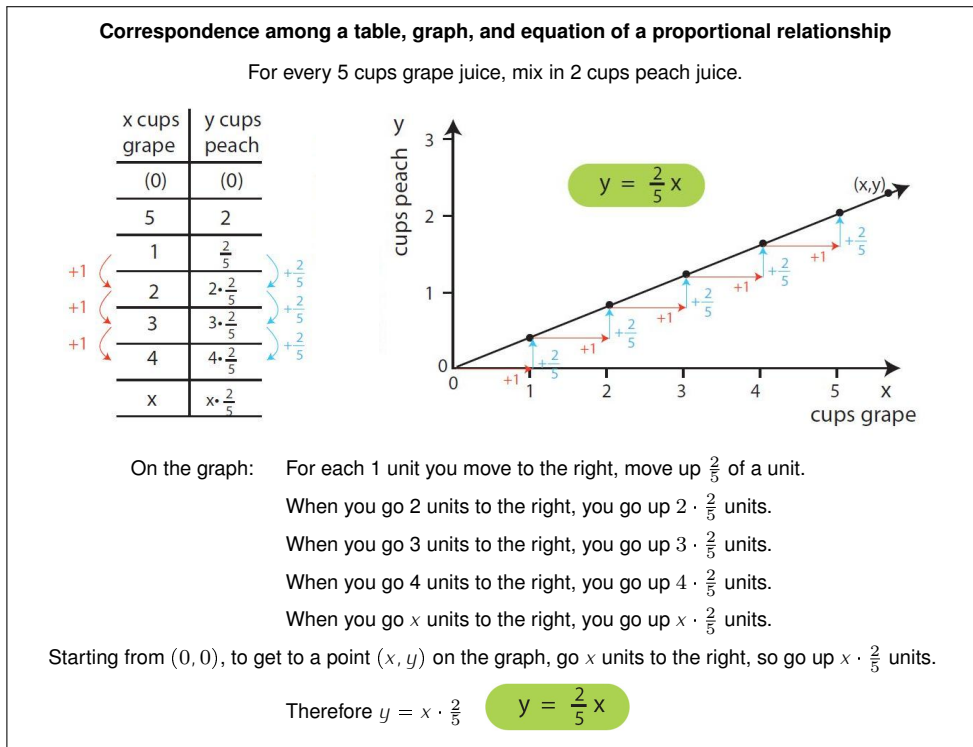
- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

7.RP.2c Represent proportional relationships by equations.

see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table and they recognize the unit rate as the vertical increase in a "unit rate triangle" or "slope triangle" with horizontal side of length 1 for a graph of a proportional relationship.^{7.RP.2b}

7.RP.2b Recognize and represent proportional relationships between quantities.

b Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.



Students connect their work with equations to their work with tables and diagrams. For example, if Seth runs 5 meters every 2 seconds, then how long will it take Seth to run 100 meters at that rate? The traditional method is to formulate an equation, $\frac{5}{2} = \frac{100}{T}$, cross-multiply, and solve the resulting equation to solve the problem. If $\frac{5}{2}$ and $\frac{100}{T}$ are viewed as unit rates obtained from the equivalent ratios $5 : 2$ and $100 : T$, then they must be equivalent fractions because equivalent ratios have the same unit rate. To see the rationale for cross-multiplying, note that when the fractions are given the common denominator $2 \cdot T$, then the numerators become $5 \cdot T$ and $2 \cdot 100$ respectively. Once the denominators are equal, the fractions are equal exactly when their numerators are equal, so $5 \cdot T$ must equal $2 \cdot 100$ for the unit rates to be equal. This is why we can solve the equation $5 \cdot T = 2 \cdot 100$ to find the amount of time it will take for Seth to run 100 meters.

A common error in setting up proportions is placing numbers in incorrect locations. This is especially easy to do when the order in which quantities are stated in the problem is switched within the problem statement. For example, the second of the following two

problem statements is more difficult than the first because of the reversal.

"If a factory produces 5 cans of dog food for every 3 cans of cat food, then when the company produces 600 cans of dog food, how many cans of cat food will it produce?"

"If a factory produces 5 cans of dog food for every 3 cans of cat food, then how many cans of cat food will the company produce when it produces 600 cans of dog food?"

Such problems can be framed in terms of proportional relationships and the constant of proportionality or unit rate, which is obscured by the traditional method of setting up proportions. For example, if Seth runs 5 meters every 2 seconds, he runs at a rate of 2.5 meters per second, so distance d (in meters) and time t (in seconds) are related by $d = 2.5t$. If $d = 100$ then $t = \frac{100}{2.5} = 40$, so he takes 40 seconds to run 100 meters.

Multistep problems Students extend their work to solving multistep ratio and percent problems.^{7.RP.3} Problems involving percent increase or percent decrease require careful attention to the referent whole. For example, consider the difference in these two percent decrease and percent increase problems:

Skateboard problem 1. After a 20% discount, the price of a SuperSick skateboard is \$140. What was the price before the discount?

Skateboard problem 2. A SuperSick skateboard costs \$140 now, but its price will go up by 20%. What will the new price be after the increase?

The solutions to these two problems are different because the 20% refers to different wholes or 100% amounts. In the first problem, the 20% is 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount. Notice that the distributive property is implicitly involved in working with percent decrease and increase. For example, in the first problem, if x is the original price of the skateboard (in dollars), then after the 20% discount, the new price is $x - 20\% \cdot x$. The distributive property shows that the new price is $80\% \cdot x$:

$$x - 20\% \cdot x = 100\% \cdot x - 20\% \cdot x = (100\% - 20\%)x = 80\% \cdot x$$

Percentages can also be used in making comparisons between two quantities. Students must attend closely to the wording of such problems to determine what the whole or 100% amount a percentage refers to.

7.RP.3 Use proportional relationships to solve multistep ratio and percent problems.

Skateboard problem 1

original 100% $5x$

20%	20%	20%	20%	20%
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discounted 80% \$140

After a 20% discount, the price is 80% of the original price. So 80% of the original is \$140.

percent	dollars
$\div 4$ (80% → \$140) $\div 4$	
or add (20% → \$35) $\cdot 5$ or add	
80%+20%	100% → \$175 $\cdot 140 \div 35$

"To find 20% I divided by 4. Then 80% plus 20% is 100%"

$x =$ original price in dollars

percent	dollars
discounted original $\frac{80}{100} = \frac{140}{x}$	
$80x = 140 \cdot 100$	
$x = \frac{140 \cdot 100}{80}$	
$= \frac{(2 \cdot 7 \cdot 2 \cdot 5)(2 \cdot 5 \cdot 10)}{2 \cdot 2 \cdot 2 \cdot 10}$	
$= 7 \cdot 5 \cdot 5$	
$= 175$	

80% of the original price is \$140.

$$\frac{80}{100} \cdot x = 140$$

$$\frac{4}{5} \cdot x = 140$$

$$x = 140 \div \frac{4}{5} = 140 \cdot \frac{5}{4} = \frac{(2 \cdot 7 \cdot 2 \cdot 5) \cdot 5}{4} = 175$$

Before the discount, the price of the skateboard was \$175.

Skateboard problem 2

original 100% \$140

20%	20%	20%	20%	20%
-----	-----	-----	-----	-----

new, increased 120% $5x$

After a 20% increase, the price is 120% of the original price. So the new price is 120% of \$140.

percent	dollars
$\div 5$ (100% → \$140) $\div 5$	
or add (20% → \$28) $\cdot 6$ or add	
100%+20%	120% → \$168 $\cdot 140 \div 28$

"To find 20% I divided by 5. Then 100% plus 20% is 120%"

$x =$ increased price in dollars

percent	dollars
discounted original $\frac{120}{100} = \frac{x}{140}$	
$\frac{12}{10} = \frac{x}{140}$	
$x = 140 \cdot \frac{12}{10} = 14 \cdot 12 = 168$	

The new, increased price is 120% of \$140.

$$x = \frac{120}{100} \cdot 140 = \frac{2 \cdot 6 \cdot 10}{2 \cdot 5 \cdot 10} \cdot 14 \cdot 2 \cdot 5 = 168$$

The new price after the increase is \$168.

Connection to Geometry One new context for proportions at Grade 7 is scale drawings.^{7.G.1} To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two different figures, or they can consider the ratio of two lengths within one figure, find a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other figure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

Connection to Statistics and Probability Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.^{7.SP.1} Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same characteristics. Therefore the ratio of the size of a portion having a certain characteristic to the size of the whole should be approximately the same for the sample as for the full population.

Where the Ratios and Proportional Relationships Progression is heading

The study of proportional relationships is a foundation for the study of functions, which continues through High School and beyond. Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.

Students extend their understanding of quantity. They write rates concisely in terms of derived units such as mi/hr rather than expressing them in terms such as " $\frac{3}{2}$ miles in every 1 hour." They encounter a wider variety of derived units and situations in which they must conceive units that measure attributes of interest.

Using percentages in comparisons

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?

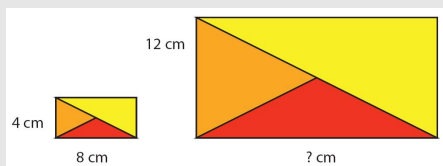
“25% more seventh graders than sixth graders means that the number of extra seventh graders is the same as 25% of the sixth graders.”

7.G.1 Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

7.SP.1 Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

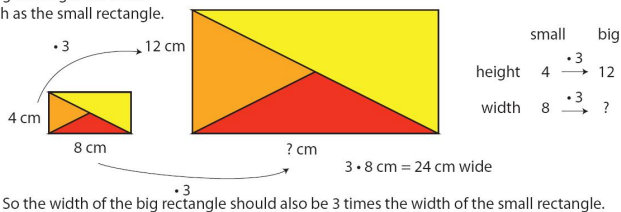
Connection to geometry

If the two rectangles are similar, then how wide is the larger rectangle?

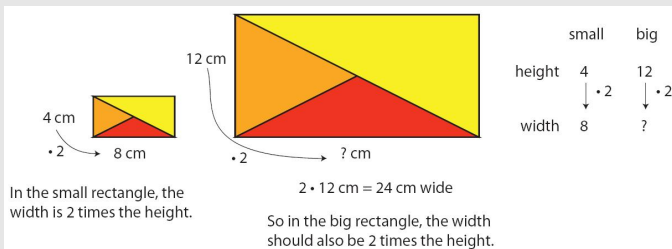


Use a scale factor: Find the scale factor from the small rectangle to the larger one:

The big rectangle is 3 times as high as the small rectangle.



Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.



Connection to statistics and probability

There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I figured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2

yellow:	3	45
blue:	7	105
total:	10	150

$\cdot 15$

"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

$$30\% \text{ yellow tiles} \quad 30\% \cdot 150 = \frac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{3}{10} \cdot 15 \cdot 10 = 45$$

$$70\% \text{ blue tiles} \quad 70\% \cdot 150 = \frac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{7}{10} \cdot 15 \cdot 10 = 105$$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."

Appendix: A framework for ratio, rate, and proportional relationships

This section presents definitions of the terms ratio, rate, and proportional relationship that are consistent with the Standards and it briefly summarizes some of the essential characteristics of these concepts. It also provides an organizing framework for these concepts. Because many different authors have used ratio and rate terminology in widely differing ways, there is a need to standardize the terminology for use with the Standards and to have a common framework for curriculum developers, professional development providers, and other education professionals to discuss the concepts. This section does not describe how the concepts should be presented to students in Grades 6 and 7.

Definitions and essential characteristics of ratios, rates, and proportional relationships

A *ratio* is a pair of non-negative numbers, $A : B$, which are not both 0.

When there are A units of one quantity for every B units of another quantity, a *rate* associated with the ratio $A : B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.) The associated *unit rate* is $\frac{A}{B}$. The term *unit rate* is the numerical part of the rate; the “unit” is used to highlight the 1 in “per 1 unit of the second quantity.” Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

A rate is expressed in terms of a unit that is derived from the units of the two quantities (such as m/s, which is derived from meters and seconds). In high school and beyond, a rate is usually written as

$$\frac{A \text{ units}}{B \text{ UNITS}}$$

where the two different fonts highlight the possibility that the quantities may have different units. In practice, when working with a ratio $A : B$, the rate $\frac{A}{B}$ units per 1 **UNIT** and the rate $\frac{B}{A}$ **UNITS** per 1 unit are both useful.

The *value* of a ratio $A : B$ is the quotient $\frac{A}{B}$ (if B is not 0). Note that the value of a ratio may be expressed as a decimal, percent, fraction, or mixed number. The value of the ratio $A : B$ tells how A and B compare multiplicatively; specifically, it tells how many times as big A is as B . In practice, when working with a ratio $A : B$, the value $\frac{A}{B}$ as well as the value $\frac{B}{A}$, associated with the ratio $B : A$, are both useful. These values of each ratio are viewed as unit rates in some contexts (see Perspective 1 in the next section).

Two ratios $A : B$ and $C : D$ are *equivalent* if there is a positive number, c , such that $C = cA$ and $D = cB$. To check that two ratios

are equivalent one can check that they have the same value (because $\frac{cA}{cB} = \frac{A}{B}$), or one can “cross-multiply” and check that $A \cdot D = B \cdot C$ (because $A \cdot cB = B \cdot cA$). Equivalent ratios have the same unit rate.

A *proportional relationship* is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs (cA, cB) , for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k , is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

Two perspectives on ratios and their associated rates in quantitative contexts

Although ratios, rates, and proportional relationships can be described in purely numerical terms, these concepts are most often used with quantities.

Ratios are often described as comparisons by division, especially when focusing on an associated rate or value of the ratio. There are also two broad categories of basic ratio situations. Some division situations, notably those involving area, can fit into either category of division. Many ratio situations can be viewed profitably from within either category of ratio. For this reason, the two categories for ratio will be described as *perspectives* on ratio.

First perspective: Ratio as a composed unit or batch Two quantities are in a ratio of A to B if for every A units present of the first quantity there are B units present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $c \cdot A$ units of the first quantity and $c \cdot B$ units of the second quantity. With this perspective, the two quantities can have the same or different units.

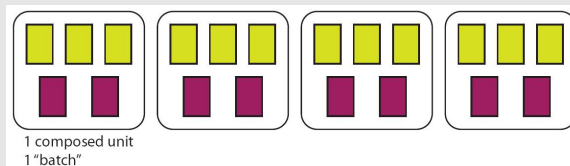
With this perspective, a ratio is specified by a composed unit or “batch,” such as “3 feet in 2 seconds,” and the unit or batch can be repeated or subdivided to create new pairs of amounts that are in the same ratio. For example, 12 feet in 8 seconds is in the ratio 3 to 2 because for every 3 feet, there are 2 seconds. Also, 12 feet in 8 seconds can be viewed as a 4 repetitions of the unit “3 feet in 2 seconds.” Similarly, $\frac{3}{2}$ feet in 1 second is $\frac{1}{2}$ of the unit “3 feet in 2 seconds.”

With this perspective, quantities that are in a ratio A to B give rise to a rate of $\frac{A}{B}$ units of the first quantity for every 1 unit of the second quantity (as well as to the rate of $\frac{B}{A}$ units of the second quantity for every 1 unit of the first quantity). For example, the ratio 3 feet in 2 seconds gives rise to the rate $\frac{3}{2}$ feet for every 1 second.

Two perspectives on ratio

- 1) There are 3 cups of apple juice for every 2 cups of grape juice in the mixture.

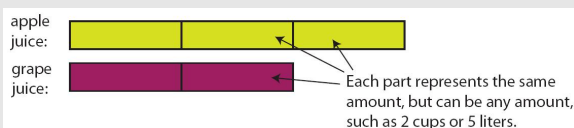
This way uses a composed unit: 3 cups apple juice and 2 cups grape juice. Any mixture that is made from some number of the composed unit is in the ratio 3 to 2.



In each of these mixtures, apple juice and grape juice are mixed in a ratio of 3 to 2:

# cups apple juice	3	6	9	12	$\frac{3}{2}$	1
# cups grape juice	2	4	6	8	1	$\frac{2}{3}$
		made of 2 composed units		made of 1/2 of a composed unit		

- 2) The mixture is made from 3 parts apple juice and 2 parts grape juice, where all parts are the same size, but can be any size.



If 1 part is :	1 cup	2 cups	5 liters	3 quarts
amt of apple juice:	3 cups	6 cups	15 liters	9 quarts
amt of grape juice:	2 cups	4 cups	10 liters	6 quarts

With this perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c , can be viewed as the numerical part of a rate associated with the ratio $A : B$.

Second perspective: Ratio as fixed numbers of parts Two quantities which have the same units, are in a ratio of A to B if there is a part of some size such that there are A parts present of the first quantity and B parts present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $A \cdot c$ units of the first quantity and $B \cdot c$ units of the second quantity.

With this perspective, one thinks of a ratio as two pieces. One piece is constituted of A parts, the other of B parts. To create pairs of measurements in the same ratio, one specifies an amount and fills each part with that amount. For example, in a ratio of 3 parts sand to 2 parts cement, each part could be filled with 5 cubic yards, so that there are 15 cubic yards of sand and 10 cubic yards of cement; or each part could be filled with 10 cubic meters, so that there are 30 cubic meters of sand and 20 cubic meters of cement. When describing a ratio from this perspective, the units need not be explicitly stated, as in "mix sand and cement in a ratio of 3 to 2." However, the type of quantity must be understood or stated explicitly, as in "by volume" or "by weight."

With this perspective, a ratio $A : B$ has an associated value, $\frac{A}{B}$, which describes how the two quantities are related multiplicatively. Specifically, $\frac{A}{B}$ is the factor that tells how many times as much of the first quantity there is as of the second quantity. (Similarly, the factor $\frac{B}{A}$ associated with the ratio $B : A$, tells how many times as much of the second quantity there is as of the first quantity.) For example, if sand and cement are mixed in a ratio of 3 to 2, then there is $\frac{3}{2}$ times as much sand as cement and there is $\frac{2}{3}$ times as much cement as sand.

With this second perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c , can be considered a factor that does not have a unit.