Progressions for the Common Core State Standards in Mathematics (draft)

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For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

Number and Operations in Base Ten, K–5

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding in computations, decomposing 1 ten into 10 ones and composing a ten from 10 ones.

The power of the base-ten system is in repeated bundling by ten: 10 tens make a unit called a hundred. Repeating this process of creating new units by bundling in groups of ten creates units called *thousand, ten thousand, hundred thousand* . . . In learning about decimals, children partition a one into 10 equal-sized smaller units, each of which is a tenth. Each base-ten unit can be understood in terms of any other base-ten unit. For example, one hundred can be viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths. Algorithms• for operations in base ten draw on such relationships among the base-ten units.

Computations Standard algorithms• for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit.

Beginning in Kindergarten, the requisite abilities develop gradually over the grades. Experience with addition and subtraction within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2 standard.[2.OA.2] Computations within 20 that "cross 10," such as $9 + 8$ or $13 - 6$, are especially relevant to NBT because they afford the development of the Level 3 make-a-ten strategies for addition and subtraction described in the OA Progression. From the NBT perspective, make-a-ten strategies are (implicitly) the first instances of composing or decomposing a base-ten unit. Such strategies are a foundation for understanding in Grade 1 that addition may require composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve decomposing a ten.[2.NBT.7]

Strategies and algorithms The Standards distinguish strategies• from algorithms. Work with computation begins with use of strategies and "efficient, accurate, and generalizable methods." (See Grade 1 critical areas 1 and 2, Grade 2 critical area 2; Grade 4 critical area 1.) For each operation, the culmination of this work is signaled in the Standards by use of the term "standard algorithm."

Initially, students compute using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction (or multiplication and division). They relate their strategies to written methods and explain the reasoning used (for addition within 100 in Grade 1; for addition and subtraction within 1000 in Grade 2) or illustrate and explain their calculations with equations, rectangular arrays, and/or area models (for multiplication and division in Grade 4).

Students' initial experiences with computation also include development, discussion, and use of "efficient, accurate, and generalizable methods." So from the beginning, students see, discuss, and explain methods that can be generalized to all numbers represented in the base-ten system. Initially, they may use written methods that include extra helping steps to record the underlying reasoning. These helping step variations can be important initially for under-

• From the Standards glossary:

> **Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. This progression gives examples of different recording methods and discusses their advantages and disadvantages.

• The Standards do not specify a particular standard algorithm for each operation. This progression gives examples of algorithms that could serve as the standard algorithm and discusses their advantages and disadvantages.

[1.OA.6] Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

[2.OA.2] Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

[1.NBT.4] Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

[2.NBT.7] Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

• From the Standards glossary:

> **Computation strategy.** Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Examples of computation strategies are given in this progression and in the Operations and Algebraic Thinking Progression.

standing. Over time, these methods can and should be abbreviated into shorter written methods compatible with fluent use of standard algorithms.

Students may also develop and discuss mental or written calculation methods that cannot be generalized to all numbers or are less efficient than other methods.

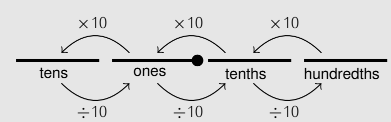
Mathematical practices The Standards for Mathematical Practice are central in supporting students' progression from understanding and use of strategies to fluency with standard algorithms. The initial focus in the Standards on understanding and explaining such calculations, with the support of visual models, affords opportunities for students to see mathematical structure as accessible, important, interesting, and useful.

Students learn to see a number as composed of its base-ten units (MP.7). They learn to use this structure and the properties of operations to reduce computing a multi-digit sum, difference, product, or quotient to a collection of single-digit computations in different base-ten units. (In some cases, the Standards refer to "multi-digit" operations rather than specifying numbers of digits. The intent is that sufficiently many digits should be used to reveal the standard algorithm for each operation in all its generality.) Repeated reasoning (MP.8) that draws on the uniformity of the base-ten system is a part of this process. For example, in addition computations students generalize the strategy of making a ten to composing 1 base-ten unit of next-highest value from 10 like base-ten units.

Students abstract quantities in a situation (MP.2) and use concrete models, drawings, and diagrams (MP.4) to help conceptualize (MP.1), solve (MP.1, MP.3), and explain (MP.3) computational problems. They explain correspondences between different methods (MP.1) and construct and critique arguments about why those methods work (MP.3). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten?, and to probe into the referents for symbols used (MP.2), e.g., does that 1 represent the number of apples in the problem?

Some methods may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, comparing methods offers opportunities to raise the topic of using appropriate tools strategically (MP.5). Comparing methods can help to illustrate the advantages of standard algorithms: standard algorithms are general methods that minimize the number of steps needed and, once, fluency is achieved, do not require new reasoning.

Uniformity of the base-ten system



For any base-ten unit, 10 copies compose 1 base-ten unit of next-highest value, e.g., 10 ones are 1 ten, 10 tens are 1 hundred, etc.

Kindergarten

In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1+9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place valueK.NBT.1 Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

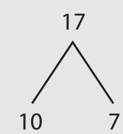
• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, . . . , nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section.

The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones).

By working with teen numbers in this way in Kindergarten, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

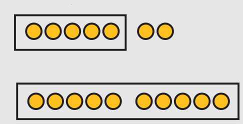
Number-bond diagram and equation



$17 = 10 + 7$

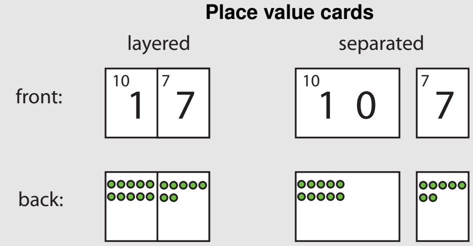
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames



Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

Place value cards



Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings. When any of the number arrangements is turned over, the one card is hidden under the tens card. Children can see this and that they need to move the ones dots above and on the right side of the tens card.

Grade 1

In first grade, students learn to view ten ones as a unit called a ten. The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and they add and subtract using this understanding.

Extend the counting sequence and understand place value Through structured learning time, discussion, and practice students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers 10, . . . , 90, in written and in spoken form, as 1 ten, . . . , 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones. Saying 67 as "6 tens, 7 ones" as well as "sixty-seven" can help students focus on the tens and ones structure of written numerals.

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols $>$, $=$, and $<$. Correctly placing the $<$ and $>$ symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.

Use place value understanding and properties of operations to add and subtract First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or

Part of a numeral list

91	101	111
92	102	112
93	103	113
94	104	114
95	105	115
96	106	116
97	107	117
98	108	118
99	109	119
100	110	120

In the classroom, a list of the numerals from 1 to 120 can be shown in columns of 10 to help highlight the base-ten structure, e.g., in the leftmost column, the 9s (indicating 9 tens) are lined up and the ones increase by 1 from 91 to 99. The numbers 101, . . . , 120 may be especially difficult for children to write.

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

a 10 can be thought of as a bundle of ten ones—called a "ten."

b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

1.NBT.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones. In particular, showing composition of a ten with objects or drawings affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

Combining tens and ones separately as illustrated in the margin can be extended to the general method of combining like base-ten units. The margin illustrates combining ones, then tens. Like base-ten units can be combined in any order, but going from smaller to larger eliminates the need to go back to a given place to add in a new unit. For example, in computing 46 + 37 by combining tens, then ones (going left to right), one needs to go back to add in the new 1 ten: "4 tens and 3 tens is 7 tens, 6 ones and 7 ones is 13 ones which is 1 ten and 3 ones, 7 tens and 1 ten is 8 tens. The total is 8 tens and 3 ones: 83."

Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings of 5-groups can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

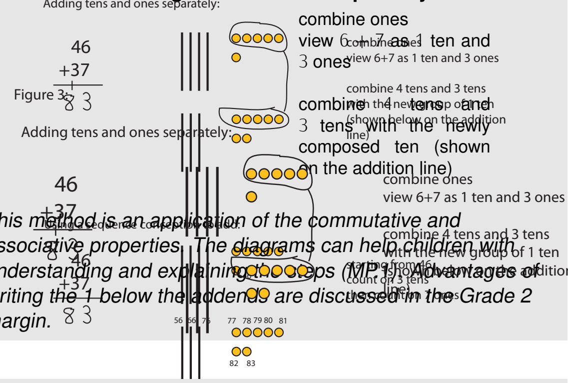
First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as 70 − 40 can be viewed as 7 tens minus 4 tens and represented with concrete models such as objects bundled in tens or drawings. Children use the relationship between subtraction and addition when they view 80 − 70 as an unknown addend addition problem, 70 + □ = 80, and reason that 1 ten must be added to 70 to make 80, so 80 − 70 = 10.

First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases. This helps students to avoid making the generalization "in each column, subtract the larger digit from the smaller digit, independent of whether the larger digit is in the subtrahend or minuend," e.g., making the error 82 - 45 = 43.

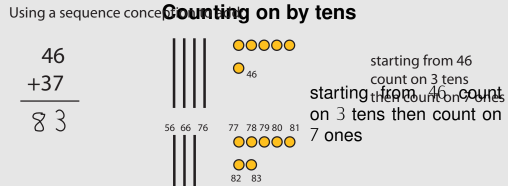
Figure 3:

Adding tens and ones separately



This method is an application of the commutative and associative properties. The diagrams can help children with understanding and explaining the steps (MP.1). Advantages of writing the 1 below the addends are discussed in the Grade 2 margin.

Counting on by tens



Counting on by tens from 46, beginning 56, 66, 76, then counting on by ones. This method can be generalized, but the complexity of the counting on required and the lack of efficiency becomes apparent as the number of digits in the addends increases.

[1.NBT.5] Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

[1.NBT.6] Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

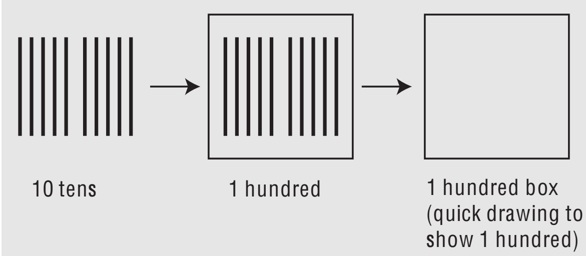
At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

[2.NBT.1a] Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a 100 can be thought of as a bundle of ten tens—called a "hundred."

Drawings to support seeing 10 tens as 1 hundred



Representations such as manipulative materials, math drawings and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3] Unlayering three-digit place value cards like the two-digit cards shown for Kindergarten and Grade 1 reveals the expanded form of the number.

[2.NBT.3] Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number said after 499 or reached after 500 counts of 1.

A major task for Grade 2 is learning the counting sequence from 100 to 1,000. As part of learning and using the base-ten structure, students count by ones within various parts of this sequence, especially the more difficult parts that "cross" tens or hundreds.

Building on their place value work, students continue to develop proficiency with mental computation.[2.NBT.8] They extend this to skip-counting by 5s, 10s, and 100s to emphasize and experience the tens and hundreds within the sequence and to prepare for multiplication.[2.NBT.2]

[2.NBT.8] Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

[2.NBT.2] Count within 1000; skip-count by 5s, 10s, and 100s.

Comparing magnitudes of two-digit numbers uses the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers uses the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4] Drawings help support these understandings.

[2.NBT.4] Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

[2.NBT.5] Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

[2.NBT.6] Add up to four two-digit numbers using strategies based on place value and properties of operations.

Use place value understanding and properties of operations to add and subtract Students fluently add and subtract within 100.[2.NBT.5] They also add and subtract within 1000.[2.NBT.7] They explain why addition and subtraction strategies work, using place value and the properties of operations, and may support their explanations with drawings or objects.[2.NBT.9] Because adding and subtracting within 100 is a special case of adding and subtracting within 1000, methods within 1000 will be discussed before fluency within 100.

Two written methods for addition within 1000 are shown in the margins of this page and the next. The first explicitly shows the hundreds, tens, and ones that are being added; this can be helpful conceptually to students. The second method, shown on the next page, explicitly shows the adding of the single digits in each place and how this approach can continue on to places on the left.

Drawings can support students in explaining these and other methods. The drawing in the margin shows addends decomposed into their base-ten units (here, hundreds, tens, and ones), with the tens and hundreds represented by quick drawings. These quick drawings show each hundred as a single unit rather than ten tens (see illustration on p. 8), generalizing the approach that students used in Grade 1 of showing a ten as a single unit rather than as 10 separate ones. The putting together of like quick drawings illustrates adding like units as specified in 2.NBT.7: add ones to ones, tens to tens, and hundreds to hundreds. The drawing also shows newly composed units. Steps of adding like units and composing new units shown in the drawing can be connected with corresponding steps in other written methods. This also facilitates discussing how different written methods may show steps in different locations or different orders (MP.1 and MP.3). The associative and the commutative properties enable adding like units to occur.

The first written method is a helping step variation that generalizes to all numbers in base ten but becomes impractical because of writing so many zeros. Students can move from this method to the second method (or another compact method) by seeing how the steps of the two methods are related. Some students might make this transition in Grade 2, some in Grade 3, but all need to make it by Grade 4 where fluency requires a more compact method.

This first method can be seen as related to oral counting-on or written adding-on methods in which an addend is decomposed into hundreds, tens, and ones. These are successively added to the other addend, with the student saying or writing successive totals. These methods require keeping track of what parts of the decomposed addend have been added, and skills of mentally counting or adding hundreds, tens, and ones correctly. For example, beginning with hundreds: 278 plus 100 is 378 ("I've used all of the hundreds"), 378 plus 30 is 408 and plus 10 (to add on all of the 40) is 418, and 418 plus 7 is 425. One way to keep track: draw the 147 and cross out parts as they are added on. Counting-on and adding-on methods become even more difficult with numbers over 1000. If they arise

[2.NBT.5] Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

[2.NBT.7] Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

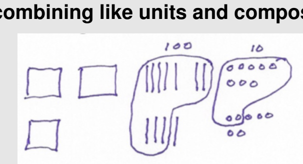
[2.NBT.9] Explain why addition and subtraction strategies work, using place value and the properties of operations.[2]

Addition: Recording newly composed units in separate rows

278	278	278	278
+ 147	+ 147	+ 147	+ 147
	300	300	300
		110	110
			15
			425

The computation shown proceeds from left to right, but could have gone from right to left. Working from left to right has two advantages: Many students prefer it because they read from left to right; working first with the largest units yields a closer approximation earlier.

Illustrating combining like units and composing new units



The drawing shows the base-ten units of 278 and 147. Like units are shown together, with boundaries drawn around ten tens and ten ones to indicate the newly composed hundred and the newly composed ten. The newly composed units could also be indicated by crossing out grouped units and drawing a single next-highest unit, e.g., crossing out the group of ten ones and drawing a single ten. Drawings like this can be used to illustrate and explain both of the written computations below.

from students, they should be discussed. But the major focus for addition within 1000 needs to be on methods such as those in the margin that are simpler for students and lead toward fluency (e.g., recording new units in separate rows shown) or are sufficient for fluency (e.g., recording new units in one row).

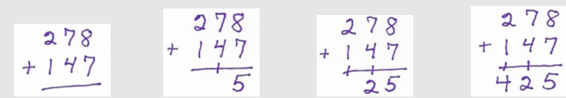
Drawings and steps for a generalizable method of subtracting within 1000 are shown in the margin. The total 425 does not have enough tens or ones to subtract the 7 tens or 8 ones in 278. Therefore one hundred is decomposed to make ten tens and one ten is decomposed to make ten ones. These decompositions can be done and written in either order; starting from the left is shown because many students prefer to operate in that order. In the middle step, one hundred has been decomposed (making 3 hundreds, 11 tens, 15 ones) so that the 2 hundreds 7 tens and 8 ones in 278 can be subtracted. These subtractions of like units can also be done in any order. When students alternate decomposing and subtracting like units, they may forget to decompose entirely or in a given column after they have just subtracted (e.g., after subtracting 8 from 15 to get 7, they move left to the tens column and see a 1 on the top and a 7 on the bottom and write 6 because they are in subtraction mode, having just subtracted the ones).

Students can also subtract within 1000 by viewing a subtraction as an unknown addend problem, e.g., $278 + ? = 425$. Counting-on and adding-on methods such as those described above for addition can be used. But as with addition, the major focus needs to be on methods that lead toward fluency or are sufficient for fluency (e.g., recording as shown in the second row in the margin).

In Grade 1, students have added within 100 using concrete models or drawings and used at least one method that is generalizable to larger numbers (such as between 101 and 1000). In Grade 2, they can make that generalization, using drawings for explanation as discussed above. This extension could be done first for two-digit numbers (e.g., $78 + 47$) so that students can see and discuss composing both ones and tens without the complexity of hundreds in the drawings or numbers (imagine the margin examples for $78 + 47$). After computing totals that compose both ones and tens for two-digit numbers, then within 1000, the type of problems required for fluency in Grade 2 seem easy, e.g., $28 + 47$ requires only composing a new ten from ones. This is now easier to do without drawings: one just records the new ten before it is added to the other tens or adds it to them mentally.

A similar approach can be taken for subtraction: first using concrete models or drawings to solve subtractions within 100 that involve decomposing one ten, then rather quickly solving subtractions that require two decompositions. Spending a long time on subtraction within 100 can stimulate students to count on or count down, which, as discussed above, are methods that are considerably more difficult with numbers above 100. Problems with different types of decompositions could be included so that students solve problems

Addition: Recording newly composed units in the same row



Add the ones, $8 + 7$, and record these 15 ones with 1 on the line in the tens column and 5 below in the ones place.

Add the tens, $7 + 4 + 1$, and record these 12 tens with 1 on the line in the hundreds column and 2 below in the tens place.

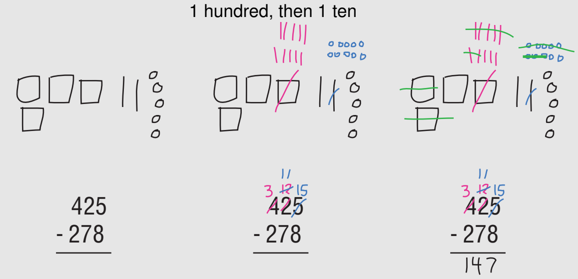
Add the hundreds, $2 + 1 + 1$, and record these 4 hundreds below in the hundreds column.

Digits representing newly composed units are placed below the addends, on the line. This placement has several advantages. Each two-digit partial sum (e.g., "15") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 5) rather than "write the 5 and carry the 1" (write 5, then 1).

Subtraction: Decomposing where needed first

decomposing left to right, 1 hundred, then 1 ten

now subtract



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

requiring two, one, and no decompositions. Then students can spend time on subtractions that include multiple hundreds (totals from 201 to 1000). Relative to these experiences, the objectives for fluency at this grade are easy: focusing within 100 just on the two cases of one decomposition (e.g., $73 - 28$) or no decomposition (e.g., $78 - 23$) without drawings.

Students also add up to four two-digit numbers using strategies based on place value and properties of operations.[2.NBT.6] This work affords opportunities for students to see that they may have to compose more than one ten, and as many as three new tens. It is also an opportunity for students to reinforce what they have learned by informally using the commutative and associative properties. They could mentally add all of the ones, then write the new tens in the tens column, and finish the computation in writing. They could successively add each addend or add the first two and last two addends and then add these totals. Carefully chosen problems could suggest strategies that depend on specific numbers. For example, $38+47+93+62$ can be easily added by adding the first and last numbers to make 100, adding the middle two numbers to make 140, and increasing 140 by 100 to make 240. Students also can develop special strategies for particularly easy computations such as $398 + 529$, where the 529 gives 2 to the 398 to make 400, leaving 400 plus 527 is 927. But the major focus in Grade 2 needs to remain on the methods that work for all numbers and generalize readily to numbers beyond 1000.

[2.NBT.6] Add up to four two-digit numbers using strategies based on place value and properties of operations.

Grade 3

At Grade 3, the major focus is multiplication,• so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

• See the progression on Operations and Algebraic Thinking.

Use place value understanding and properties of operations to perform multi-digit arithmetic Students fluently add and subtract within 1000 using methods based on place value, properties of operations, and/or the relationship between addition and subtraction.[3.NBT.2] They focus on methods that generalize readily to larger numbers so that these methods can be extended to 1,000,000 in Grade 4 and fluency can be reached with such larger numbers. Fluency within 1000 implies that students use written methods without concrete models or drawings, though concrete models or drawings can be used with explanations to overcome errors and to continue to build understanding as needed.

[3.NBT.2] Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students and often is sufficient for practical purposes. Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

[3.NBT.1] Use place value understanding to round whole numbers to the nearest 10 or 100.

The special role of 10 in the base-ten system is important in understanding multiplication of one-digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication: $3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150$. It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non-zero digits, then shift the product one place to the left to make the result ten times as large.

[3.NBT.3] Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

Grade 4

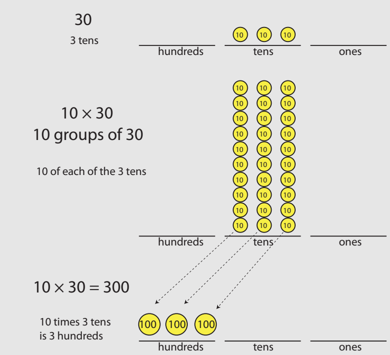
At Grade 4, students extend their work in the base-ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi-digit whole numbers In the base-ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (thousand, million, billion, trillion, etc.). Thus, 457,000 is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in previous grades apply to these numbers, because of the uniformity of the base-ten system.[4.NBT.3]

[4.NBT.1] Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 represented as 3 tens each taken 10 times

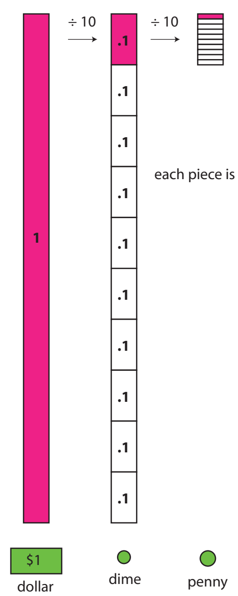


Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

[4.NBT.2] Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

[4.NBT.3] Use place value understanding to round multi-digit whole numbers to any place.

Decimal notation and fractions Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base-ten system to non-whole numbers.



Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$. As with whole-number expansions in the base-ten system, each unit in this decomposition is ten times the unit to its right, reflecting the uniformity of the base-ten system. This can be connected with the use of base-ten notation to represent $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

[4.NF.5] Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[3]

[4.NF.6] Use decimal notation for fractions with denominators 10 or 100.

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place, as illustrated in the margin.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins 3.1415 . . .)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

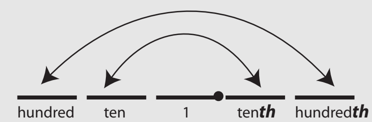
Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students understand how 2 tenths and 7 hundredths make 27 hundredths. Place value cards can be layered with the places farthest from the decimal point on the bottom (see illustration of the whole number cards on p. 5). These places are then covered by each place toward the decimal point: Tenths go on top of hundredth, and tens go on top of hundreds (for example, .2 goes on top of .07 to make .27, and 20 goes on top of 700 to make 720).

Use place value understanding and properties of operations to perform multi-digit arithmetic Students fluently add and subtract multi-digit numbers through 1,000,000 using the standard algorithm.[4.NBT.4] Because students in Grade 2 and Grade 3 have been using at least one method that readily generalizes to 1,000,000, this extension does not have to take a long time. Thus, students will have time for the major NBT focus for this grade: multiplication and division.

In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose, which is why 4.NBT.5 explicitly states that they are to be used to illustrate and explain the calculation. By reasoning repeatedly (MP.8) about the connection between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

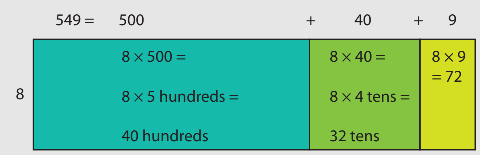
Symmetry with respect to the ones place



[4.NBT.4] Fluently add and subtract multi-digit whole numbers using the standard algorithm.

[4.NBT.5] Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Multiplication: Illustrating partial products with an area model

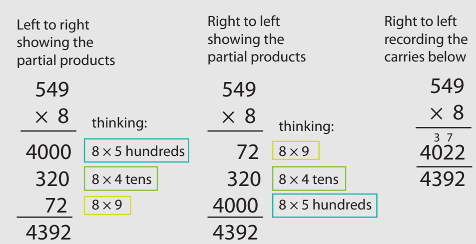


Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

An area model can be used for any multiplication situation after students have discussed how to show an equal groups or a compare situation with an area model by making the length of the rectangle represent the size of the equal groups or the larger compared quantity imagining things inside the square units to make an array (but not drawing them), and understanding that the dimensions of the rectangle are the same as the dimensions of the imagined array, e.g., an array illustrating 8 x 549 would have 8 rows and 549 columns. (See the Operations and Algebraic Thinking Progression for discussion of "equal groups" and "compare" situations.)

Multiplication: Recording methods



The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549*, thus keeping the digits of the products close to each other, e.g., the* 7 *from* $8 \times 9 = 72$ *is written diagonally to the left of the* 2 *rather than above the* 4 *in* 549*. The colors indicate correspondences with the area model above.*

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

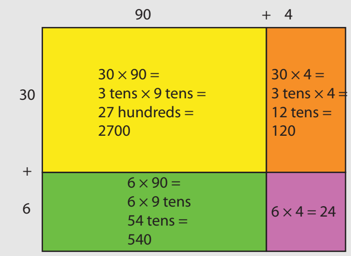
Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$\begin{aligned} 36 \times 94 &= (30 + 6) \times (94) \\ &= 30 \times 94 + 6 \times 94 \\ &= 30 \times (90 + 4) + 6 \times (90 + 4) \\ &= 30 \times 90 + 30 \times 4 + 6 \times 90 + 6 \times 4. \end{aligned}$$

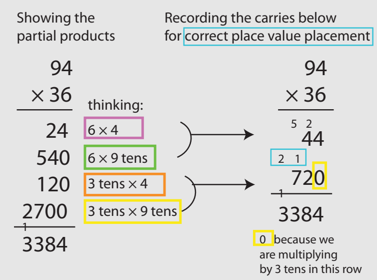
The four products in the last line correspond to the four rectangles in the area model in the margin. Their factors correspond to the factors in written methods. When written methods are abbreviated, some students have trouble seeing how the single-digit factors are related to the two-digit numbers whose product is being computed (MP.2). They may find it helpful initially to write each two-digit number as the sum of its base-ten units (e.g., writing next to the calculation $94 = 90 + 4$ and $36 = 30 + 6$) so that they see what the single digits are. Some students also initially find it helpful to write what they are multiplying in front of the partial products (e.g., $6 \times 4 = 24$). These helping steps can be dropped when they are no longer needed. At any point before or after their acquisition of fluency, some students may prefer to multiply from the left because they find it easier to align the subsequent products under this biggest product.

Illustrating partial products with an area model



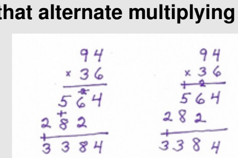
The products of base-ten units are shown as parts of a rectangular region. Such area models can support understanding and explaining of different ways to record multiplication. For students who struggle with the spatial demands of other methods, a useful helping step method is to make a quick sketch like this with the lengths labeled and just the partial products, then to add the partial products outside the rectangle.

Methods that compute partial products first



These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94*. The digits* 2 *and* 1 *are surrounded by a blue box. The* 1 *from* $30 \times 4 = 120$ *is placed correctly in the hundreds place and the digit* 2 *from* $30 \times 90 = 2700$ *is placed correctly in the thousands place. If these digits had been placed above* 94*, they would be in incorrect places. Note that the* 0 *(surrounded by a yellow box) in the ones place of the second row of the method on the right is there because the whole row of digits is produced by multiplying by* 30 *(not* 3*). Colors on the left correspond with the area model above.*

Methods that alternate multiplying and adding



These methods put the newly composed units from a partial product in the correct column, then they are added to the next partial product. These alternating methods are more difficult than the methods above that show the four partial products. The first method can be used in Grade 5 division when multiplying a partial quotient times a two-digit divisor.

Not shown is the recording method in which the newly composed units are written above the top factor (e.g., 94). This puts the hundreds digit of the tens times ones product in the tens column (e.g., the 1 hundred in 120 from 30×4 *above the 9 tens in 94). This placement violates the convention that students have learned: a digit in the tens place represents tens, not hundreds.*

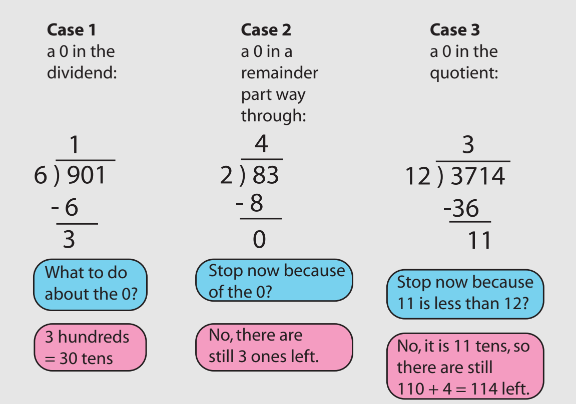
General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. See the figure in the margin. As with multiplication, this relies on the distributive property. This can be viewed as finding the side length of a rectangle (the divisor is the length of the other side) or as allocating objects (the divisor is the number of groups or the number of objects in each group). See the figure on the next page for an example.

Multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$.• In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation.

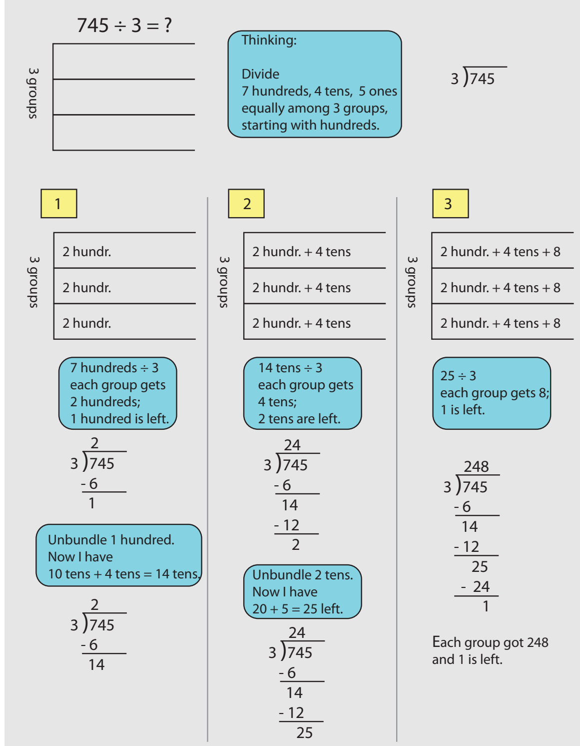
Cases involving 0 in division may require special attention. See the figure below.

Cases involving 0 in division



[4.NBT.6] Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Division as finding group size



$745 \div 3$ can be viewed as allocating 745 objects bundled in 7 hundreds, 4 tens, and 3 ones equally among 3 groups. In Step 1, the 2 indicates that each group got 2 hundreds, the 6 is the number of hundreds allocated, and the 1 is the number of hundreds not allocated. After Step 1, the remaining hundred is decomposed as 10 tens and combined with the 4 tens (in 745) to make 14 tens.

• A note on notation

The result of division within the system of whole numbers is frequently written as:

$$84 \div 10 = 8\,\text{R}\,4 \quad\text{and}\quad 44 \div 5 = 8\,\text{R}\,4.$$

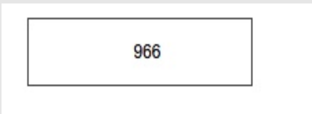
Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation $8\,\text{R}\,4$ does not indicate a number.

Rather than writing the result of division in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written as:

$$84 = 8 \times 10 + 4 \quad\text{and}\quad 44 = 8 \times 5 + 4.$$

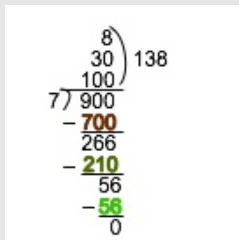
Division as finding side length

? hundreds + ? tens + ? ones

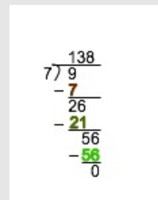


Find the unknown length of the rectangle; first find the hundreds, then the tens, then the ones.

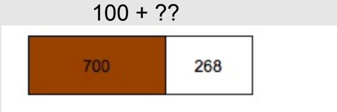
Method A



Method B



100 + ??

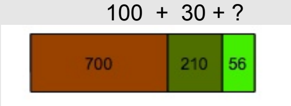


The length has 1 hundred, making a rectangle with area 700.

Method A records the difference of the areas as 966 – 700 = 266, showing the remaining area (266). Only hundreds are subtracted; the tens and ones digits do not change.

Method B records only the hundreds digit (2) of the difference and "brings down" the unchanged tens digit (6). These digits represent: 2 hundreds + 6 tens = 26 tens.

100 + 30 + ?

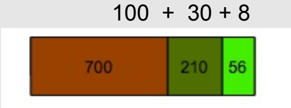


The length has 3 tens, making a rectangle with area 210.

Method A records the difference of the areas as 266 – 210 = 56. Only hundreds and tens are subtracted; the ones digit does not change.

Method B records only the tens digit (5) of the difference and "brings down" the ones digit (6). These digits represent: 5 tens + 6 ones = 56 ones.

100 + 30 + 8



The length has 8 ones, making an area of 56. The original rectangle can now be seen as composed of three smaller rectangles with areas of the amounts that were subtracted from 966.

Method A shows each partial quotient and has the final step adding them (going from 100 + 30 + 8 to 138).

Method B abbreviates these partial quotients. These can be said explicitly when explaining the method (e.g., 7 hundreds subtracted from the 9 hundreds is 2 hundreds).

966 ÷ 7 can be viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The divisor, partial quotients (100, 30, 8), and final quotient (138) represent quantities in length units and the dividend represents a quantity in area units.

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare, and how numbers round for decimals to thousandths.

New at Grade 5 is the use of whole number exponents to denote powers of 10.[5.NBT.2] Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right. So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole number and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value. Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

[5.NBT.2] Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Perform operations with multi-digit whole numbers and with decimals to hundredths At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

Division in Grade 5 extends Grade 4 methods to two-digit divisors.[5.NBT.6] Students continue to decompose the dividend into base-ten units and find the quotient place by place, starting from the highest place. They illustrate and explain their calculations using equations, rectangular arrays, and/or area models. Estimating the quotients is a new aspect of dividing by a two-digit number. Even if students round the dividend appropriately, the resulting estimate may need to be adjusted up or down. Sometimes multiplying the ones of a two-digit divisor composes a new thousand, hundred, or ten. These newly composed units can be written as part of the division computation, added mentally, or as part of a separate multiplication computation. Students who need to write decomposed units when subtracting need to remember to leave space to do so.

[5.NBT.5] Fluently multiply multi-digit whole numbers using the standard algorithm.

[5.NBT.6] Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Recording division after an underestimate

1655 ÷ 27	1
	10 } 61
Rounding 27	(30) 50
to 30 produces	27) 1655
the underestimate	-1350
50 at the first step	305
but this method	-270
allows the division	35
process to be	-27
continued	8

Because of the uniformity of the structure of the base-ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.[5.NBT.7] Like base-ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. A whole number is not usually written with a decimal point, but a decimal point followed by one or more 0s can be inserted on the right (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base-ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing newly composed units on the addition line, and they can decompose units wherever needed before subtracting.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single-digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base-ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.[5.NF.3] When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.[5.NF.4] Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation-based method is not reliable in all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place

[5.NBT.7] Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths to make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns, then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

[5.NF.7b] Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Extending beyond Grade 5

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. Students use these ideas again when they calculate with polynomials in high school.

The distributive property and like units: Multiplication of whole numbers and polynomials

$52 \cdot 73$	$(5x+2)(7x+3)$	
$= (5 \cdot 10 + 2)(7 \cdot 10 + 3)$	$= (5x+2)(7x+3)$	decomposing as like units (powers of 10 or powers of x)
$= 5 \cdot 10(7 \cdot 10 + 3) + 2 \cdot (7 \cdot 10 + 3)$	$= 5x(7x+3) + 2(7x+3)$	using the distributive property
$= 35 \cdot 10^2 + 15 \cdot 10 + 14 \cdot 10 + 2 \cdot 3$	$= 35x^2 + 15x + 14x + 2 \cdot 3$	using the distributive property again
$= 35 \cdot 10^2 + 29 \cdot 10 + 6$	$= 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)