# Progressions for the Common Core State Standards in Mathematics (draft)



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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

# Chapter 1

# Geometry, K–6

## Overview

Like core knowledge of number, core geometrical knowledge appears to be a universal capability of the human mind. Geometric and spatial thinking are important in and of themselves, because they connect mathematics with the physical world, and play an important role in modeling phenomena whose origins are not necessarily physical, for example, as networks or graphs. They are also important because they support the development of number and arithmetic concepts and skills. Thus, geometry is essential for all grade levels for many reasons: its mathematical content, its roles in physical sciences, engineering, and many other subjects, and its strong aesthetic connections.

This progression discusses the most important goals for elementary geometry according to three categories.

- Geometric shapes, their components (e.g., sides, angles, faces), their properties, and their categorization based on those properties.
- Composing and decomposing geometric shapes.
- Spatial relations and spatial structuring.

*Geometric shapes, components, and properties.* Students develop through a series of levels of geometric and spatial thinking. As with all of the domains discussed in the Progressions, this development depends on instructional experiences. Initially, students cannot reliably distinguish between examples and nonexamples of categories of shapes, such as triangles, rectangles, and squares.• With experience, they progress to the next level of thinking, recognizing shapes in ways that are visual or syncretic (a fusion of differing systems). At this level, students can recognize shapes as wholes, but cannot form mathematically-constrained mental images of them. A given figure is a rectangle, for example, because "it looks like a door." They do not explicitly think about the components or

• In formal mathematics, a geometric shape is a boundary of a region, e.g., "circle" is the boundary of a disk. This distinction is not expected in elementary school.

about the defining attributes, or properties, of shapes. Students then move to a descriptive level in which they can think about the components of shapes, such as triangles having three sides. For example, kindergartners can decide whether all of the sides of a shape are straight and they can count the sides. They also can discuss if the shape is closed• and thus convince themselves that a three-sided shape is a triangle even if it is "very skinny" (e.g., an isosceles triangle with large obtuse angle).

At the analytic level, students recognize and characterize shapes by their *properties*.[1] For instance, a student might think of a square as a figure that has four equal sides and four right angles. Different components of shapes are the focus at different grades, for instance, second graders measure lengths and fourth graders measure angles (see the Geometric Measurement Progression). Students find that some combinations of properties signal certain classes of figures and some do not; thus the seeds of geometric implication are planted. However, only at the next level, abstraction, do students see relationships between classes of figures (e.g., understand that a square is a rectangle because it has all the properties of rectangles).• Competence at this level affords the learning of higher-level geometry, including deductive arguments and proof.

Thus, learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years.

*Composing and decomposing.* As with their learning of shapes, components, and properties, students follow a progression to learn about the composition and decomposition of shapes. Initially, they lack competence in composing geometric shapes. With experience, they gain abilities to combine shapes into pictures–first, through trial and error, then gradually using attributes. Finally, they are able to synthesize combinations of shapes into new shapes.•

Students compose new shapes by putting two or more shapes together and discuss the shapes involved as the parts and the totals. They decompose shapes in two ways. They take away a part by covering the total with a part (for example, covering the "top" of a

[1] In this progression, the term "property" is reserved for those attributes that indicate a relationship between components of shapes. Thus, "having parallel sides" or "having all sides of equal lengths" are properties. "Attributes" and "features" are used interchangeably to indicate any characteristic of a shape, including properties, and other defining characteristics (e.g., straight sides) and nondefining characteristics (e.g., "right-side up").

• A shape with straight sides is closed if exactly two sides meet at every vertex, every side meets exactly two other sides, and no sides cross each other.

**Levels of geometric thinking**

*Visual/syncretic.* Students recognize shapes, e.g., a rectangle "looks like a door."

*Descriptive.* Students perceive properties of shapes, e.g., a rectangle has four sides, all its sides are straight, opposite sides have equal length.

*Analytic.* Students characterize shapes by their properties, e.g., a rectangle has opposite sides of equal length and four right angles.

*Abstract.* Students understand that a rectangle is a parallelogram because it has all the properties of parallelograms.

• Note that in the U.S., that the term "trapezoid" may have two different meanings. In their study *The Classification of Quadrilaterals* (Information Age Publishing, 2008), Usiskin et al. call these the exclusive and inclusive definitions:

T(E): a trapezoid is a quadrilateral with exactly one pair of parallel sides

T(I): a trapezoid is a quadrilateral with at least one pair of parallel sides.

These different meanings result in different classifications at the analytic level. According to T(E), a parallelogram is not a trapezoid; according to T(I), a parallelogram is a trapezoid.

Both definitions are legitimate. However, Usiskin et al. conclude, "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-bound geometry books, to favor the inclusive definition."

• **A note about research** The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis. Additionally, there is suggestive evidence that this type of composition corresponds with, and may support, children's ability to compose and decompose numbers.

triangle with a smaller triangle to make a trapezoid). And they take shapes apart by building a copy beside the original shape to see what shapes that shape can be decomposed into (initially, they may need to make the decomposition on top of the total shape). With experience, students are able to use a composed shape as a new unit in making other shapes. Grade 1 students make and use such a unit of units (for example, making a square or a rectangle from two identical right triangles, then making pictures or patterns with such squares or rectangles). Grade 2 students make and use three levels of units (making an isosceles triangle from two 1″ by 2″ right triangles, then making a rhombus from two of such isosceles triangles, and then using such a rhombus with other shapes to make a picture or a pattern). Grade 2 students also compose with two such units of units (for example, making adjacent strips from a shorter parallelogram made from a 1″ by 2″ rectangle and two right triangles and a longer parallelogram made from a 1″ by 3″ parallelogram and the same two right triangles). Grade 1 students also rearrange a composite shape to make a related shape, for example, they change a 1″ by 2″ rectangle made from two right triangles into an isosceles triangle by flipping one right triangle. They explore such rearrangements of the two right triangles more systematically by matching the short right angle side (a tall isosceles triangle and a parallelogram with a "little slant"), then the long right angle sides (a short isosceles triangle and a parallelogram with a "long slant"). Grade 2 students rearrange more complex shapes, for example, changing a parallelogram made from a rectangle and two right triangles into a trapezoid by flipping one of the right triangles to make a longer and a shorter parallel side.

Composing and decomposing requires and thus builds experience with properties such as having equal lengths or equal angles.

*Spatial structuring and spatial relations*. Early composition and decomposition of shape is a foundation for spatial structuring, an important case of geometric composition and decomposition. Students need to conceptually structure an array to understand two-dimensional objects and sets of such objects in two-dimensional space as truly two-dimensional. Such spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because it takes previously abstracted items as content and integrates them to form new structures. For two-dimensional arrays, students must see a composite of squares (iterated units) and as a composite of rows or columns (units of units). Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane. Spatial relations such as above/below and right/left are understood within such spatial structures. These understandings begin informally, later becoming more formal.

The ability to structure a two-dimensional rectangular region into rows and columns of squares requires extended experiences with shapes derived from squares (e.g., squares, rectangles, and right triangles) and with arrays of contiguous squares that form patterns. Development of this ability benefits from experience with compositions, decompositions, and iterations of the two, but it requires extensive experience with arrays.

Students make pictures from shapes whose sides or points touch, and they fill in outline puzzles. These gradually become more elaborate, and students build mental visualizations that enable them to move from trial and error rotating of a shape to planning the orientation and moving the shape as it moves toward the target location. Rows and columns are important units of units within square arrays for the initial study of area, and squares of 1 by 1, 1 by 10, and 10 by 10 are the units, units of units, and units of units of units used in area models of two-digit multiplication in Grade 4. Layers of three-dimensional shapes are central for studying volume in Grade 5. Each layer of a right rectangular prism can also be structured in rows and columns, such layers can also be viewed as units of units of units. That is, as 1000 is a unit (one thousand) of units (one hundred) of units (tens) of units (singletons), a right rectangular prism can be considered a unit (solid, or three-dimensional array) of units (layers) of units (rows) of units (unit cubes).

*Summary*. The Standards for Kindergarten, Grade 1, and Grade 2 focus on three major aspects of geometry. Students build understandings of shapes and their properties, becoming able to do and discuss increasingly elaborate compositions, decompositions, and iterations of the two, as well as spatial structures and relations. In Grade 2, students begin the formal study of measure, learning to use units of length and use and understand rulers. Measurement of angles and parallelism are a focus in Grades 3, 4, and 5. At Grade 3, students begin to consider relationships of shape categories, considering two levels of subcategories (e.g., rectangles are parallelograms and squares are rectangles). They complete this categorization in Grade 5 with all necessary levels of categories and with the understanding that any property of a category also applies to all shapes in any of its subcategories. They understand that some categories overlap (e.g., not all parallelograms are rectangles) and some are disjoint (e.g., no square is a triangle), and they connect these with their understanding of categories and subcategories. Spatial structuring for two- and three-dimensional regions is used to understand what it means to measure area and volume of the simplest shapes in those dimensions: rectangles at Grade 3 and right rectangular prisms at Grade 5 (see the Geometric Measurement Progression).

K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/“corners”) and other attributes (e.g., having sides of equal length).

## Kindergarten

Understanding and describing shapes and space is one of the two critical areas of Kindergarten mathematics. Students develop geometric concepts and spatial reasoning from experience with two perspectives on space: the shapes of objects and the relative positions of objects.

In the domain of shape, students learn to match two-dimensional shapes even when the shapes have different orientations.[K.G.4] They learn to name shapes such as circles, triangles, and squares, whose names occur in everyday language, and distinguish them from nonexamples of these categories, often based initially on visual prototypes. For example, they can distinguish the most typical examples of triangles from the obvious nonexamples.

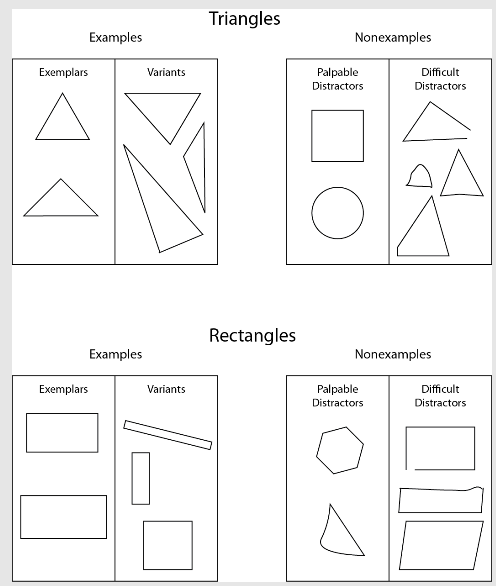
From experiences with varied examples of these shapes (e.g., the variants shown in the margin), students extend their initial intuitions to increasingly comprehensive and accurate intuitive concept images of each shape category.[•] These richer concept images support students' ability to perceive a variety of shapes in their environments and describe these shapes in their own words.[MP7] This includes recognizing and informally naming three-dimensional shapes, e.g., "balls," "boxes," "cans." Such learning might also occur in the context of solving problems that arise in construction of block buildings and in drawing pictures, simple maps, and so forth.

Students then refine their informal language by learning mathematical concepts and vocabulary so as to increasingly describe their physical world from geometric perspectives, e.g., shape, orientation, spatial relations (MP4). They increase their knowledge of a variety of shapes, including circles, triangles, squares, rectangles, and special cases of other shapes such as regular hexagons, and trapezoids with unequal bases and non-parallel sides of equal length.[K.G.1] [•] They learn to sort shapes according to these categories.[MP7] The need to explain their decisions about shape names or classifications prompts students to attend to and describe certain features of the shapes.[K.G.4] That is, concept images and names they have learned for the shapes are the raw material from which they can abstract common features.[MP2] This also supports their learning to represent shapes informally with drawings and by building them from components (e.g., manipulatives such as sticks).[K.G.5] With repeated experiences such as these, students become more precise (MP6). They begin to attend to attributes, such as being a triangle, square, or rectangle, and being *closed* figures with *straight* sides. Similarly, they attend to the lengths of sides and, in simple situations, the size of angles when comparing shapes.

Students also begin to name and describe three-dimensional shapes with mathematical vocabulary, such as "sphere," "cube," "cylinder," and "cone."[K.G.1] They identify faces of three-dimensional shapes as two-dimensional geometric figures[K.G.4] and explicitly identify shapes as two-dimensional ("flat" or lying in a plane) or three-dimensional



K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).




*Exemplars* are the typical visual prototypes of the shape category.

*Variants* are other examples of the shape category.

*Palpable distractors* are nonexamples with little or no overall resemblance to the exemplars.

*Difficult distractors* are visually similar to examples but lack at least one defining attribute.

• Tall and Vinner describe *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (See "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics, 12*, pp. 151–169.) This term was formulated by Shlomo Vinner in 1980.

MP7 Mathematically proficient students look closely to discern a pattern or structure.

K.G.1 Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

• If the exclusive definition of trapezoid is used (see p. 3), such trapezoids would be called isosceles trapezoids.

MP7 Young students, for example, . . . may sort a collection of shapes according to how many sides the shapes have.

MP2 Mathematically proficient students have the ability to abstract a given situation.

K.G.5 Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

("solid").[K.G.3]

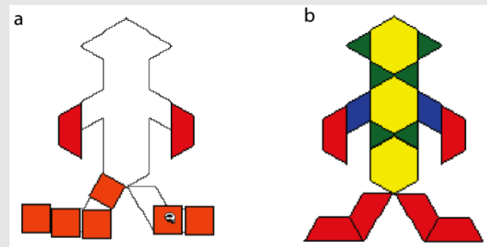
A second important area for kindergartners is the composition of geometric figures. Students not only build shapes from components, but also compose shapes to build pictures and designs. Initially lacking competence in composing geometric shapes, they gain abilities to combine shapes–first by trial and error and gradually by considering components–into pictures. At first, side length is the only component considered. Later experience brings an intuitive appreciation of angle size.

Students combine two-dimensional shapes and solve problems such as deciding which piece will fit into a space in a puzzle, intuitively using geometric motions (slides, flips, and turns, the informal names for translations, reflections, and rotations, respectively). They can construct their own outline puzzles and exchange them, solving each other's.

Finally, in the domain of spatial reasoning, students discuss not only shape and orientation, but also the relative positions of objects, using terms such as "above," "below," "next to," "behind," "in front of," and "beside."[K.G.1] They use these spatial reasoning competencies, along with their growing knowledge of three-dimensional shapes and their ability to compose them, to model objects in their environment, e.g., building a simple representation of the classroom using unit blocks and/or other solids (MP4).

[K.G.3] Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

**Combining shapes to build pictures**




Students first use trial and error (part a) and gradually consider components (part b).

## Grade 1

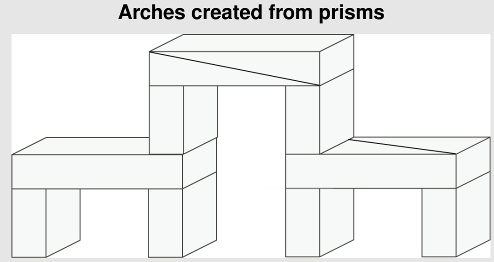
In Grade 1, students reason about shapes. They describe and classify shapes, including drawings, manipulatives, and physical-world objects, in terms of their geometric *attributes*. That is, based on early work recognizing, naming, sorting, and building shapes from components, they describe in their own words why a shape belongs to a given category, such as squares, triangles, circles, rectangles, rhombuses, (regular) hexagons, and trapezoids (with bases of different lengths and nonparallel sides of the same length). In doing so, they differentiate between geometrically defining attributes (e.g., "hexagons have six straight sides") and nondefining attributes (e.g., color, overall size, or orientation).[1.G.1] For example, they might say of this shape, "This has to go with the squares, because all four sides are the same, and these are square corners. It doesn't matter which way it's turned" (MP3, MP7). They explain why the variants shown earlier (p. 6) are members of familiar shape categories and why the difficult distractors are not, and they draw examples and nonexamples of the shape categories. Students learn to sort shapes accurately and exhaustively based on these attributes, describing the similarities and differences of these familiar shapes and shape categories (MP7, MP8).

[1.G.1] Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) ; build and draw shapes to possess defining attributes.

From the early beginnings of informally matching shapes and solving simple shape puzzles, students learn to intentionally compose and decompose plane and solid figures (e.g., putting two congruent isosceles triangles together with the explicit purpose of making a rhombus),[1.G.2] building understanding of part-whole relationships as well as the properties of the original and composite shapes. In this way, they learn to perceive a combination of shapes as a single new shape (e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles). Thus, they develop competencies that include solving shape puzzles and constructing designs with shapes, creating and maintaining a shape as a unit, and combining shapes to create composite shapes that are conceptualized as independent entities (MP2). They then learn to substitute one composite shape for another congruent composite composed of different parts.

[1.G.2] Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[2]

**Arches created from prisms**



Right rectangular prisms are composed with prisms with right triangle bases. Note that the dimensions of the triangular prism on the top arch differ from the dimensions of that on the right.

Students build these competencies, often more slowly, in the domain of three-dimensional shapes. For example, students may intentionally combine two right triangular prisms to create a right rectangular prism, and recognize that each triangular prism is half of the rectangular prism.[1.G.3] They also show recognition of the composite shape of "arch." (Note that the process of combining shapes to create a composite shape is much like combining 10 ones to make 1 ten.) Even simple compositions, such as building a floor or wall of rectangular prisms, build a foundation for later mathematics.

[1.G.3] Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

As students combine shapes, they continue to develop their sophistication in describing geometric attributes and properties and determining how shapes are alike and different, building foundations for measurement and initial understandings of properties such as congruence and symmetry. Students can learn to use their intuitive understandings of measurement, congruence, and symmetry to guide their work on tasks such as solving puzzles and making simple origami constructions by folding paper to make a given two- or three-dimensional shape (MP1).•

• For example, students might fold a square of paper once to make a triangle or nonsquare rectangle. For examples of other simple two- and three-dimensional origami constructions, see http://www.origami-instructions.com/simple-origami.html.

## Grade 2

Students learn to name and describe the defining attributes of categories of two-dimensional shapes, including circles, triangles, squares, rectangles, rhombuses, trapezoids, and the general category of quadrilateral. They describe pentagons, hexagons, septagons, octagons, and other polygons by the number of sides, for example, describing a septagon as either a "seven-gon" or simply "seven-sided shape" (MP2).[2.G.1] Because they have developed both verbal descriptions of these categories and their defining attributes and a rich store of associated mental images, they are able to draw shapes with specified attributes, such as a shape with five sides or a shape with six angles.[2.G.1] They can represent these shapes' attributes accurately (within the constraints of fine motor skills). They use length to identify the properties of shapes (e.g., a specific figure is a rhombus because all four of its sides have equal length). They recognize right angles, and can explain the distinction between a rectangle and a parallelogram without right angles and with sides of different lengths (sometimes called a "rhomboid").

[2.G.1] Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[3] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

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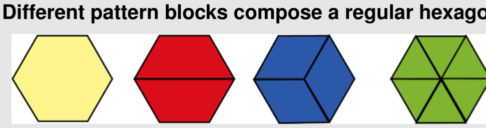
Students learn to combine their composition and decomposition competencies to build and operate on composite units (units of units), intentionally substituting arrangements or composites of smaller shapes or substituting several larger shapes for many smaller shapes, using geometric knowledge and spatial reasoning to develop foundations for area, fraction, and proportion. For example, they build the same shape from different parts, e.g., making with pattern blocks, a regular hexagon from two trapezoids, three rhombuses, or six equilateral triangles. They recognize that the hexagonal faces of these constructions have equal area, that each trapezoid has half of that area, and each rhombus has a third of that area.[2.G.3]

**Different pattern blocks compose a regular hexagon**

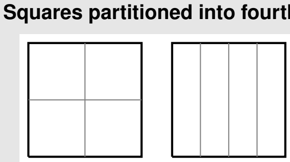


[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

This example emphasizes the fraction concepts that are developed; students can build and recognize more difficult composite shapes and solve puzzles with numerous pieces. For example, a tangram is a special set of 7 shapes which compose an isosceles right triangle. The tangram pieces can be used to make many different configurations and tangram puzzles are often posed by showing pictures of these configurations as silhouettes or outlines. These pictures often are made more difficult by orienting the shapes so that the sides of right angles are not parallel to the edges of the page on which they are displayed. Such pictures often do not show a grid that shows the composing shapes and are generally not preceded by analysis of the composing shapes.

Students also explore decompositions of shapes into regions that are congruent or have equal area.[2.G.3] For example, two squares can be partitioned into fourths in different ways. Any of these fourths represents an equal share of the shape (e.g., "the same amount of cake") even though they have different shapes.

**Squares partitioned into fourths**



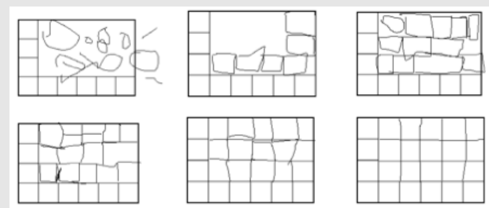
These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Another type of composition and decomposition is essential to students' mathematical development—*spatial structuring*. Students need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP7).

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

**Levels of thinking in spatial structuring**



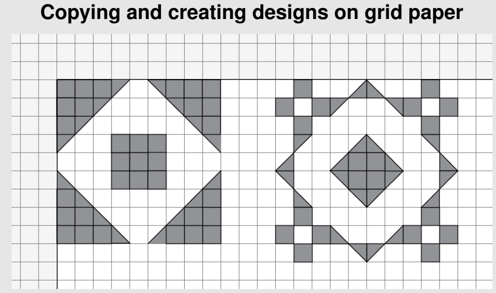
Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because previously abstracted items (e.g., squares, including composites made up of squares) are used as the content of new mental structures. Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. At first, students might tile a rectangle with identical squares or draw such arrays and then count the number of squares one-by-one. In the lowest levels of the progression, they may even lose count of or double-count some squares. As the mental structuring process helps them organize their counting, they become more systematic, using the array structure to guide the quantification. Eventually, they begin to use repeated addition of the number in each row or each column. Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane.

Foundational activities, such as forming arrays by tiling a rectangle with identical squares (as in building a floor or wall from blocks) should have developed students' basic spatial structuring competencies before second grade—if not, teachers should ensure that their students learn these skills. Spatial structuring can be further developed with several activities with grids. Games such as "battleship" can be useful in this regard.

Another useful type of instructional activity is copying and creating designs on grids. Students can copy designs drawn on grid paper by placing manipulative squares and right triangles onto other copies of the grid. They can also create their own designs, draw their creations on grid paper, and exchange them, copying each others' designs.

**Copying and creating designs on grid paper**

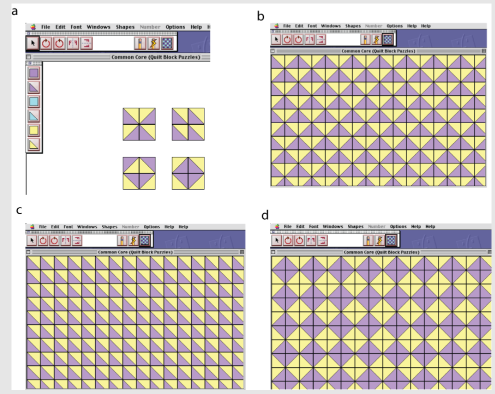


Students can copy designs such as these, using only squares (all of the same size) and isosceles right triangles (half of the square) as manipulatives, creating their copies on paper with grid squares of the same size as the manipulative square.

Another, more complex, activity designing tessellations by iterating a "core square." Students design a unit composed of smaller units: a core square composed of a 2 by 2 array of squares filled with square or right triangular regions. They then create the tessellation ("quilt") by iterating that core in the plane. This builds spatial structuring because students are iterating "units of units" and reflecting on the resulting structures. Computer software can aid in this iteration.

These various types of composition and decomposition experiences simultaneously develop students' visualization skills, including recognizing, applying, and anticipating (MP1) the effects of applying rigid motions (slides, flips, and turns) to two-dimensional shapes.

**"Core squares" iterated to make a tessellation**



In the software environment illustrated above (Pattern Blocks and Mini-Quilts software), students need to be explicitly aware of the transformations they are using in order to use slide, flip, and turn tools. At any time, they can tessellate any one of the core squares using the "quilt" tool indicated by the rightmost icon. Part a shows four different core squares. The upper left core square produces the tessellation in part b. Parts c and d are produced, respectively, by the upper right and lower right core squares. Interesting discussions result when the class asks which designs are mathematically different (e.g., should a rotation or flip of the core be counted as "different"s?).

## Grade 3

Students analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3).[3.G.1] They explicitly relate and combine these classifications. Because they have built a firm foundation of several shape categories, these categories can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories. A description of these categories of quadrilaterals is illustrated in the margin. The Standards do not require that such representations be constructed by Grade 3 students, but they should be able to draw examples of quadrilaterals that are not in the subcategories.

Similarly, students learn to draw shapes with prespecified attributes, without making a priori assumptions regarding their classification.[MP1] For example, they could solve the problem of making a shape with two long sides of the same length and two short sides of the same length that is not a rectangle.

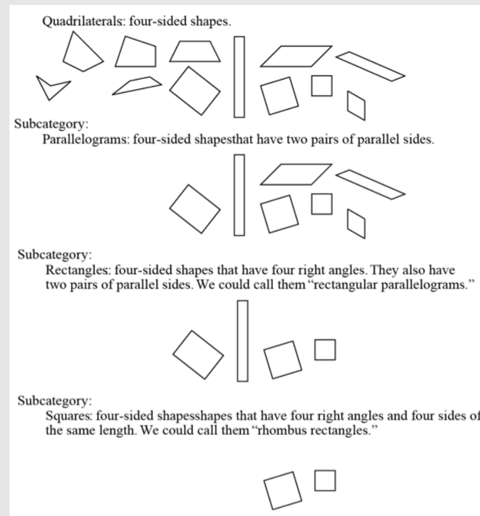
Students investigate, describe, and reason about decomposing and composing polygons to make other polygons. Problems such as finding all the possible different compositions of a set of shapes involve geometric problem solving and notions of congruence and symmetry (MP7). They also involve the practices of making and testing conjectures (MP1), and convincing others that conjectures are correct (or not) (MP3). Such problems can be posed even for sets of simple shapes such as tetrominoes, four squares arranged to form a shape so that every square shares at least one side and sides coincide or share only a vertex.

More advanced paper-folding (origami) tasks afford the same mathematical practices of seeing and using structure, conjecturing, and justifying conjectures. Paper folding can also illustrate many geometric concepts. For example, folding a piece of paper creates a line segment. Folding a square of paper twice, horizontal edge to horizontal edge, then vertical edge to vertical edge, creates a right angle, which can be unfolded to show four right angles. Students can be challenged to find ways to fold paper into rectangles or squares and to explain why the shapes belong in those categories.

Students also develop more competence in the composition and decomposition of rectangular regions, that is, spatially structuring rectangular arrays. They learn to partition a rectangle into identical squares[3.G.2] by anticipating the final structure and thus forming the array by drawing rows and columns (see the bottom right example on p. 11; some students may still need work building or drawing squares inside the rectangle first). They count by the number of columns or rows, or use multiplication to determine the number of

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

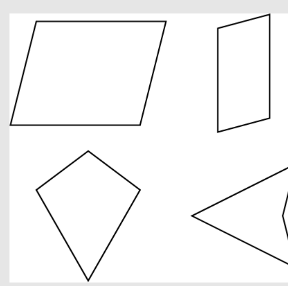
**Quadrilaterals and some special kinds of quadrilaterals**




The representations above might be used by teachers in class. Note that the left-most four shapes in the first section at the top left have four sides but do not have properties that would place them in any of the other categories shown (parallelograms, rectangles, squares).

[MP1] Students . . . analyze givens, constraints, relationships, and goals.

**Quadrilaterals that are not rectangles**



These quadrilaterals have two pairs of sides of the same length but are not rectangles. A kite is on lower left and a deltoid is at lower right.

[3.G.2] Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

squares in the array. They also learn to rotate these arrays physically and mentally to view them as composed of smaller arrays, allowing illustrations of properties of multiplication (e.g., the commutative property and the distributive property).

## Grade 4

Students describe, analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3), including explicit use of angle sizes[4.G.1] and the related geometric properties of perpendicularity and parallelism.[4.G.2] They can identify these properties in two-dimensional figures. They can use side length to classify triangles as equilateral, equiangular, isosceles, or scalene; and can use angle size to classify them as acute, right, or obtuse. They then learn to cross-classify, for example, naming a shape as a right isosceles triangle. Thus, students develop explicit awareness of and vocabulary for many concepts they have been developing, including points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Such mathematical terms are useful in communicating geometric ideas, but more important is that constructing examples of these concepts, such as drawing angles and triangles that are acute, obtuse, and right,[4.G.1] help students form richer concept images connected to verbal definitions. That is, students have more complete and accurate mental images and associated vocabulary for geometric ideas (e.g., they understand that angles can be larger than 90° and their concept images for angles include many images of such obtuse angles). Similarly, students see points and lines as abstract objects: Lines are infinite in extent and points have location but no dimension. Grids are made of points and lines and do not end at the edge of the paper.

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

Students also learn to apply these concepts in varied contexts (MP4). For example, they learn to represent angles that occur in various contexts as two rays, explicitly including the reference line, e.g., a horizontal or vertical line when considering slope or a "line of sight" in turn contexts. They understand the size of the angle as a rotation of a ray on the reference line to a line depicting slope or as the "line of sight" in computer environments. Students might solve problems of drawing shapes with turtle geometry.• Analyzing the shapes in order to construct them (MP1) requires students to explicitly formulate their ideas about the shapes (MP4, MP6). For instance, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such experiences help students connect what are often initially isolated ideas about the concept of angle.

• The computer programming language Logo has a pointer, often a icon of a turtle, that draws representations of points, line segments, and shapes, with commands such as "forward 100" and "right 120."

Students might explore line segments, lengths, perpendicularity, and parallelism on different types of grids, such as rectangular and triangular (isometric) grids (MP1, MP2).[4.G.2, 4.G.3] Can you find a non-rectangular parallelogram on a rectangular grid? Can you find a rectangle on a triangular grid? Given a segment on a rectangular grid that is not parallel to a grid line, draw a parallel segment of the same length with a given endpoint. Given a half of a figure and

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

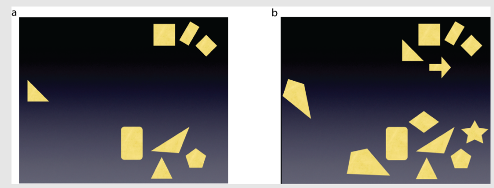
a line of symmetry, can you accurately draw the other half to create a symmetric figure?

Students also learn to reason about these concepts. For example, in "guess my rule" activities, they may be shown two sets of shapes and asked where a new shape belongs (MP1, MP2).[4.G.2]

In an interdisciplinary lesson (that includes science and engineering ideas as well as items from mathematics), students might encounter another property that all triangles have: rigidity. If four fingers (both thumbs and index fingers) form a shape (keeping the fingers all straight), the shape of that quadrilateral can be easily changed by changing the angles. However, using three fingers (e.g., a thumb on one hand and the index and third finger of the other hand), students can see that the shape is fixed by the side lengths. Triangle rigidity explains why this shape is found so frequently in bridge, high-wire towers, amusement park rides, and other constructions where stability is sought.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

**Guess My Rule**




Students can be shown the two groups of shapes in part a and asked "Where does the shape on the left belong?" They might surmise that it belongs with the other triangles at the bottom. When the teacher moves it to the top, students must search for a different rule that fits all the cases.

Later (part b), students may induce the rule: "Shapes with at least one right angle are at the top." Students with rich visual images of right angles and good visualization skills would conclude that the shape at the left (even though it looks vaguely like another one already at the bottom) has one right angle, thus belongs at the top.

## Grade 5

By the end of Grade 5, competencies in shape composition and decomposition, and especially the special case of spatial structuring of rectangular arrays (recall p. 11), should be highly developed (MP7). Students need to develop these competencies because they form a foundation for understanding multiplication, area, volume, and the coordinate plane. To solve area problems, for example, the ability to decompose and compose shapes plays multiple roles. First, students understand that the area of a shape (in square units) is the number of unit squares it takes to cover the shape without gaps or overlaps. They also use decomposition in other ways. For example, to calculate the area of an "L-shaped" region, students might decompose the region into rectangular regions, then decompose each region into an array of unit squares, spatially structuring each array into rows or columns. Students extend their spatial structuring in two ways. They learn to spatially structure in three dimensions; for example, they can decompose a right rectangular prism built from cubes into layers, seeing each layer as an array of cubes. They use this understanding to find the volumes of right rectangular prisms with edges whose lengths are whole numbers as the number of unit cubes that pack the prisms (see the Geometric Measurement Progression). Second, students extend their knowledge of the coordinate plane, understanding the continuous nature of two-dimensional space and the role of fractions in specifying locations in that space.

Thus, spatial structuring underlies coordinates for the plane as well, and students learn both to apply it and to distinguish the objects that are structured. For example, they learn to interpret the components of a rectangular grid structure as line segments or lines (rather than regions) and understand the precision of location that these lines require, rather than treating them as fuzzy boundaries or indicators of intervals. Students learn to reconstruct the levels of counting and quantification that they had already constructed in the domain of discrete objects to the coordination of (at first) two continuous linear measures. That is, they learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions.[5.G.1]

Although students can often "locate a point," these understandings are beyond simple skills. For example, initially, students often fail to distinguish between two different ways of viewing the point $(2, 3)$, say, as instructions: "right 2, up 3"; and as the point defined by being a distance 2 from the $y$-axis and a distance 3 from the $x$-axis. In these two descriptions the 2 is first associated with the $x$-axis, then with the $y$-axis.

They connect ordered pairs of (whole number) coordinates to points on the grid, so that these coordinate pairs constitute numerical objects and ultimately can be operated upon as single mathematical entities. Students solve mathematical and real-world problems using coordinates. For example, they plan to draw a symmetric fig-

5.G.1 Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., $x$-axis and $x$-coordinate, $y$-axis and $y$-coordinate).

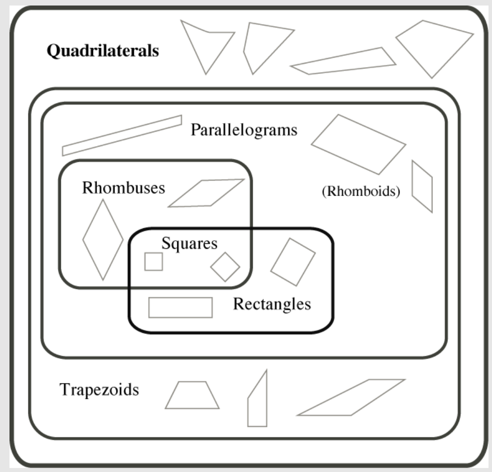
ure using computer software in which students' input coordinates that are then connected by line segments.[5.G.2]

Students learn to analyze and relate categories of two-dimensional and three-dimensional shapes explicitly based on their properties.[5.G.4] Based on analysis of properties, they classify two-dimensional figures in hierarchies. For example, they conclude that all rectangles are parallelograms, because they are all quadrilaterals with two pairs of opposite, parallel, equal-length sides (MP3). In this way, they relate certain categories of shapes as subclasses of other categories.[5.G.3] This leads to understanding propagation of properties; for example, students understand that squares possess all properties of rhombuses and of rectangles. Therefore, if they then show that rhombuses' diagonals are perpendicular bisectors of one another, they infer that squares' diagonals are perpendicular bisectors of one another as well.

5.G.2 Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

5.G.4 Classify two-dimensional figures in a hierarchy based on properties.

**Venn diagram showing classification of quadrilaterals**




Note that rhomboids are parallelograms that are not rhombuses or rectangles. This example uses the inclusive definition of trapezoid (see p. [pageref "T(E)"]).

5.G.3 Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

## Grade 6

Problems involving areas and volumes extend previous work and provide a context for developing and using equations.[6.G.1, 6.G.2] Students' competencies in shape composition and decomposition, especially with spatial structuring of rectangular arrays (recall p. 11), should be highly developed. These competencies form a foundation for understanding multiplication, formulas for area and volume, and the coordinate plane.[6.NS.6, 6.NS.8]

Using the shape composition and decomposition skills acquired in earlier grades, students learn to develop area formulas for parallelograms, then triangles. They learn how to address three different cases for triangles: a height that is a side of a right angle, a height that "lies over the base" and a height that is outside the triangle.[MP.1]

Through such activity, students learn that that any side of a triangle can be considered as a base and the choice of base determines the height (thus, the base is not necessarily horizontal and the height is not always in the interior of the triangle). The ability to view a triangle as part of a parallelogram composed of two copies of that triangle and the understanding that area is additive (see the Geometric Measurement Progression) provides a justification (MP3) for halving the product of the base times the height, helping students guard against the common error of forgetting to take half.

Also building on their knowledge of composition and decomposition, students decompose rectilinear polygons into rectangles, and decompose special quadrilaterals and other polygons into triangles and other shapes, using such decompositions to determine their areas, and justifying and finding relationships among the formulas for the areas of different polygons.

Building on the knowledge of volume (see the Geometric Measurement Progression) and spatial structuring abilities developed in earlier grades, students learn to find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism.[6.G.2 MP.1 MP.4]

Students also analyze and compose and decompose polyhedral solids. They describe the shapes of the faces, as well as the number of faces, edges, and vertices. They make and use drawings of solid shapes and learn that solid shapes have an outer surface as well as an interior. They develop visualization skills connected to their mathematical concepts as they recognize the existence of, and visualize, components of three-dimensional shapes that are not visible from a given viewpoint (MP1). They measure the attributes of these shapes, allowing them to apply area formulas to solve surface area problems (MP7). They solve problems that require them to distinguish between units used to measure volume and units used to measure area (or length). They learn to plan the construction of

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

MP.1 Students . . . try special cases and simpler forms of the original problem in order to gain insight into its solution.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

MP.1 explain correspondences

MP.4 write an equation to describe a situation.

complex three-dimensional compositions through the creation of corresponding two-dimensional nets (e.g., through a process of digital fabrication and/or graph paper).[6.G.4] For example, they may design a living quarters (e.g., a space station) consistent with given specifications for surface area and volume (MP2, MP7). In this and many other contexts, students learn to apply these strategies and formulas for areas and volumes to the solution of real-world and mathematical problems.[6.G.1, 6.G.2] These problems include those in which areas or volumes are to be found from lengths or lengths are to be found from volumes or areas and lengths.

Students extend their understanding of properties of two-dimensional shapes to use of coordinate systems.[6.G.3] For example, they may specify coordinates for a polygon with specific properties, justifying the attribution of those properties through reference to relationships among the coordinates (e.g., justifying that a shape is a parallelogram by computing the lengths of its pairs of horizontal and vertical sides).

As a precursor for learning to describe cross-sections of three-dimensional figures,[7.G.3] students use drawings and physical models to learn to identify parallel lines in three-dimensional shapes, as well as lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.

6.G.4 Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.G.3 Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

7.G.3 Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

### Where the Geometry Progression is Heading

Composition and decomposition of shapes is used throughout geometry from Grade 6 to high school and beyond. Compositions and decompositions of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems that involve complex shapes. Decompositions are often indicated in geometric diagrams by an auxiliary line, and using the strategy of drawing an auxiliary line to solve a problem are part of looking for and making use of structure (MP7). Recognizing the significance of an existing line in a figure is also part of looking for and making use of structure. This may involve identifying the length of an associated line segment, which in turn may rely on students' abilities to identify relationships of line segments and angles in the figure. These abilities become more sophisticated as students gain more experience in geometry. In Grade 7, this experience includes making scale drawings of geometric figures and solving problems involving angle measure, surface area, and volume (which builds on understandings described in the Geometric Measurement Progression as well as the ability to compose and decompose figures).