Progressions for the Common Core State Standards in Mathematics (draft)

24 March 2016

Suggested citation:

Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

Chapter 1

Geometry, 7–8, High School

Overview

Geometry has two important streams that begin in elementary grades: understanding properties of geometric figures and the logical connections between them, and developing and using formulas to compute lengths, areas and volumes. A third stream, coordinate geometry, surfaces in Grade 5, gains importance in Grades 6–8, and mingles with algebra to become analytic geometry in high school.

Properties of geometric figures The first stream starts with learning in K–5 about geometric shapes, culminating in their classification in Grade 5. In Grade 6, students develop an informal understanding of congruence as they dissect figures in order to calculate their areas. An important principle in this work is that if two figures match exactly when they are put on top of each other (the informal notion of congruence) then they have the same area. In Grade 7, students gain an informal notion of similarity as they work with scale drawings. They draw—or try to draw—geometric shapes that obey given conditions, acquiring experience that they use in considering congruence in Grade 8 and congruence criteria in high school. Grade 8 students work with transformations—mappings of the plane to itself—understanding rigid motions and their properties from hands-on experience, then understanding congruence in terms of rigid motions. High school students analyze transformations that include dilations, understanding similarity in terms of rigid motions and dilations. Students prove theorems, using the properties of rigid motions established in Grade 8 and the properties of dilations established in high school. (Note the analogues between Grade 8 and high school standards in the table below.) This approach allows K–5 work with shapes and later work with their motions be connected

*This document does not treat in detail all of the geometry studied in Grades 7–8 and high school. Rather it gives key connections among standards and notes important pedagogical choices to be made.

to the to more abstract work of high school geometry and provides a foundation for the theorems that students prove.

Grade 6	Grade 7	Grade 8	High School
Solve real-world and mathematical problems involving area, surface area, and volume.	**Draw, construct, and describe geometrical figures and describe the relationships between them.**	**Understand congruence and similarity using physical models, transparencies, or geometry software.**	
6.G.1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes. . . .	7.G.1. Solve problems involving scale drawings of geometric figures. . . . 7.G.2. Draw . . . geometric shapes with given conditions. . . .	8.G.2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations. . . .	**Understand congruence in terms of rigid motions**
		8.G.4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations. . . .	**Understand similarity in terms of similarity transformations**
		8.G.1. Verify experimentally the properties of rotations, reflections, and translations. . . .	G-SRT.1. Verify experimentally the properties of dilations given by a center and a scale factor. . . .

Geometric measurement The second stream develops in conjunction with number and operations in Grades K–5. Students build on their experience with length measurement to understand fractions as subdivided length units on the number line. Starting in Grade 3 students work with the connection between multiplication and area, expanding to volume in Grade 5. In Grades 6–8, students apply geometric measurement to real-world and mathematical problems, making use of properties of figures as they dissect and rearrange them in order to calculate or estimate lengths, areas, and volumes. Use of geometric measurement continues in high school. Students examine it more closely, giving informal arguments to explain formulas used in earlier grades. These arguments draw on the abilities they have developed in earlier grades: dissecting and rearranging two- and three-dimensional figures; and visualizing cross-sections of three-dimensional figures.

Grade 6	Grade 7	Grade 8	High School
Solve real-world and mathematical problems involving:			**Explain volume formulas and use them to solve problems.**
area, surface area, and volume.	**angle measure, area, surface area, and volume.**	**volume of cylinders, cones, and spheres.**	
6.G.1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes.	7.G.4. Know the formulas for the area and circumference of a circle; give an informal derivation of the relationship between the circumference and area of a circle.	8.G.9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.	G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
6.G.2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. . . .	7.G.6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.	**Understand and apply the Pythagorean Theorem.** 8.G.1. Explain a proof of the Pythagorean Theorem and its converse.	
	Draw, construct, and describe geometrical figures and describe the relationships between them. 7.G.3. Describe the two-dimensional figures that result from slicing three-dimensional figures. . . .		

Analytic geometry Grade 5 also sees the first trickle of a stream that becomes important in high school, connecting geometry with algebra, by plotting pairs of non-negative integers in the coordinate plane. In Grade 6, the coordinate plane is extended to all four quadrants, in Grades 6–8 it is used to graph relationships between quantities, and in Grade 8 students use the Pythagorean Theorem to compute distances between points. Students gain further experience with the coordinate plane in high school, graphing and analyzing a variety of relationships (see the Modeling, Statistics and Probability, and Functions Progressions). They express geometric properties with equations and use coordinates to prove geometric theorems algebraically.

	Standard or group heading	Notable connections
6	**Solve real-world and mathematical problems involving area, surface area, and volume.** 6.G.3. Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these. . . .	• Make tables of equivalent ratios relating quantities and plot the pairs of values on the coordinate plane. (6.RP.3.a) • Represent points on the line and in the plane with negative number coordinates. (6.NS.6)
7		• Recognize and represent proportional relationships between quantities. Decide whether two quantities are in a proportional relationship, e.g., by graphing. Identify constant of proportionality from graph. Explain what a point on the graph of a proportional relationship means in terms of the situation. (7.RP.2)
8	**Understand and apply the Pythagorean Theorem.** 8.G.8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.	• Graph proportional relationships. Use similar triangles to explain uniqueness of slope. Solve pairs of simultaneous linear equations. (8.EE) • Determine rate of change and initial value from a graph. Interpret the rate of change and initial value of a linear function in terms of its graph. Analyze and sketch graphs. (8.F) • Investigate patterns of association in bivariate data using scatter plots and linear models. (8.SP)
HS	**Expressing Geometric Properties with Equations (G-GPE)** **Translate between the geometric description and the equation for a conic section.** **Use coordinates to prove simple geometric theorems algebraically.**	• Functions • Modeling • Statistics and Probability

Grade 7

Draw, construct, and describe geometrical figures and describe the relationships between them By sketching geometric shapes that obey given conditions,[7.G.2] students lay the foundation for the concepts of congruence and similarity in Grade 8, and for the practice of geometric deduction that will grow in importance throughout the rest of their school careers.

For example, given three side lengths, perhaps in the form of physical or virtual rods, students try to construct a triangle. Two important possibilities arise: there is no triangle or there is exactly one triangle. By examining many situations where there is no triangle, students can identify the culprit: one side that is longer than the other two put together. From this they can reason that in a triangle the sum of any two sides must be greater than the third.

The second possibility is that there is exactly one triangle.• From this students gain an intuitive notion of rigidity: the same triangle is forced on you no matter where you start to draw it. Students might wonder whether two triangles that are reflections of each other are considered the the same or different, noting that if a flip is allowed as one of the motions in superposition then the two triangles are considered the same.

Students should also work with figures with more than three sides. For example, they can contrast the rigidity of triangles with the floppiness of quadrilaterals, where it is possible to construct many different quadrilaterals with the same side lengths.

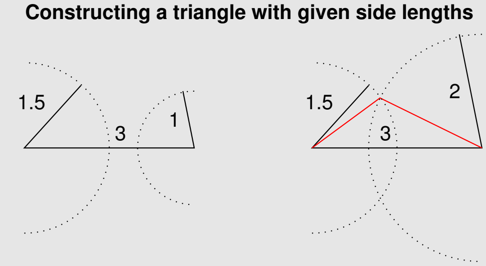
Students examine situations where they are given two sides and an angle of a triangle, or two angles and a side, preparing for the congruence criteria for triangles in high school. Implicitly, the idea of being given a side depends on what it means for two line segments to be the same. In Grade 7, it means having the same length. Using a compass to show how a line segment can be translated from one position to another can be a useful transition from the Grade 7 view of "sameness" to the Grade 8 notion of congruence.

When students are given three angles of a triangle there are also rich opportunities for discovery and reasoning. By setting lines at specified angles, either physically or virtually, they can see that the third angle is determined once two angles are given, paving the way for an understanding of the geometric force of the angle-sum theorem in Grade 8, as opposed to thinking of it as a merely numerical fact about the sum of the angles. When a triangle with given angles does exist, students can see that many such triangles exist. For example, they can translate one of the lines back and forth relative to the other two in such a way that the angles do not change (see margin), obtaining triangles that will also be viewed in high school as the results of dilations.

Students also work with scale drawings,[7.G.1] drawings that represent measurements of a real object in terms of a smaller unit of measurement. Examples of scale drawings include architectural plans,

[7.G.2] Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

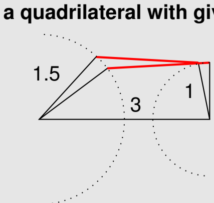
Constructing a triangle with given side lengths



It is not possible to construct a triangle with side lengths 1, 1.5, and 3. No matter how you move the smaller sides around at the ends of the largest side they will never meet, because $1 + 1.5 < 3$. If you increase the 1 to 2, you can create a triangle by finding the intersection of circles as shown.

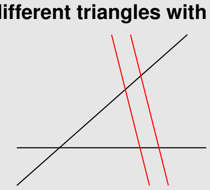
• What does "exactly one" mean? In Grade 7, two triangles with the same side lengths are considered the same if one can be moved on top of the other, so that they match exactly. In Grade 8, the movement will be described in terms of rigid motions.

Constructing a quadrilateral with given side lengths



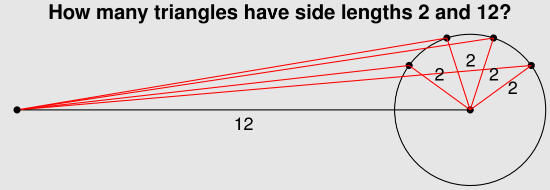
The base is fixed and the two sides are of fixed length as they move around circles centered at ends of the base. The top is a rigid rod of fixed length that moves with its endpoints on the circles, creating many quadrilaterals with the same side lengths.

Constructing different triangles with the same angles



The red lines are parallel, constructing different triangles with the black lines, all of which have the same angles.

How many triangles have side lengths 2 and 12?



[7.G.1] Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

photocopies, and many maps. Some maps, e.g., the Mercator projection of the Earth, distort distances or land areas and are thus not scale drawings. Likewise, technical drawings and photographs of three-dimensional objects that require a distortion in scale and are not scale drawings. Three-dimensional objects can be represented without distortion by scale models such as doll houses, model trains, architectural models, and souvenirs.

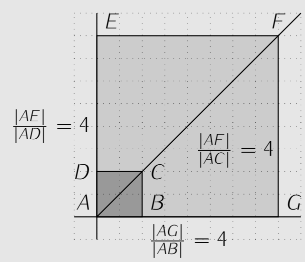
Students compute or estimate lengths in the real object by computing or measuring lengths in the drawing and multiplying by the *scale factor*. They investigate: What is the same and what is different about the scale drawings and their original counterparts? Angles in a scale drawing are the same as the corresponding angles in the real object. Lengths are not the same, but differ by a constant scale factor.

Area in the scale drawing is also a constant multiple of area in the original; however the constant is the square of the scale factor (see margin).

Students study three-dimensional figures, in particular polyhedral figures such as cubes or pyramids, and visualize them using their knowledge of two-dimensional figures.[7.G.3] A *plane section* of a three-dimensional object is a two-dimensional slice formed by an intersection of the object with a plane. Students investigate the two-dimensional figures that arise from plane sections of cubes and pyramids. In addition to developing spatial sense and visualization techniques, discussion of plane sections includes ways of generating three-dimensional objects from two-dimensional ones, paving the way for calculating their volumes in Grade 8 and high school.

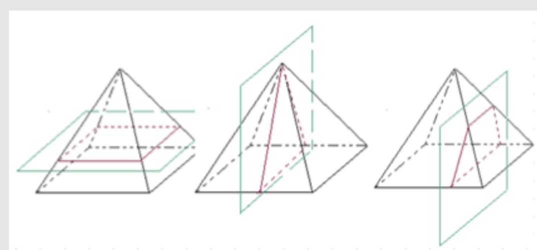
[7.G.3] Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

How area changes under scaling



The quotient of corresponding lengths is 4, while the quotient of areas is $16 = 4^2$. See also the discussion of http://www.illustrativemathematics.org/illustrations/107

Cross-sections of a pyramid



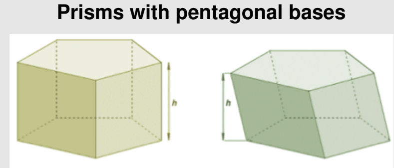
Cross-section is another name for a plane section, but often that name is reserved for a section of a three-dimensional object that is parallel to one of its planes of symmetry or perpendicular to one of its lines of symmetry. So, for example, for a cube, one line of symmetry joins the centers of opposite faces. A cross-section perpendicular to that line is a square, as is the cross-section of the right rectangular pyramid shown above left.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume In Grade 7, students extend the use of geometric terms and definitions with which they have become familiar: polygons, perimeter, area, volume and surface area of two-dimensional and three-dimensional objects, etc. They continue to apply their knowledge in order to solve problems.[7.G.6] In Grade 6, students found the area of a polygon by decomposing it into triangles and rectangles whose areas they could calculate, making use of structure (MP.7) in order to reduce the original problem to collections of simpler problems (MP.1). Now they apply the same sort of reasoning to three-dimensional figures, dissecting them in order to calculate their volumes.

[7.G.6] Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

In order to reason about the volume of a *prism* it helps to know what a prism is. Start with two planes in space that are parallel. For any polygon in one plane move it in a direction perpendicular to that plane until it reaches the other plane. The resulting three-dimensional figure is called a *right rectangular prism*, and the original polygon the *base* of the prism. (The margin also shows an oblique prism, in which the base is moved in a direction that is not perpendicular to the bottom plane.) Notice that any cross-section

Prisms with pentagonal bases

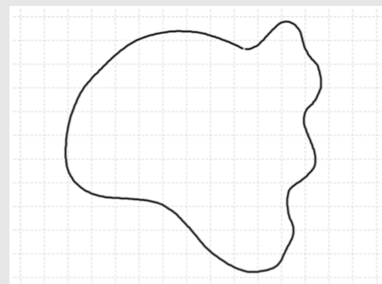


of such a figure cut by a plane parallel to the original planes is a copy of the base.

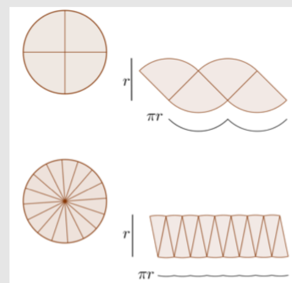
Students have long been familiar with circles and now they undertake a calculation of their perimeters and areas.[7.G.4] This is a step forward from their previous methods of calculating area by decomposing figures into rectangles and triangles. Students must now grapple with the meaning of the area of a figure with curved boundary. The area can be estimated by superimposing a square grid and counting squares inside the figure, with the estimate becoming more and more accurate as the grid is made finer and finer.

[7.G.4]Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Using a grid to estimate area



Area and circumference of a circle of radius 1



Dissecting a circle of radius 1 into smaller and smaller sectors gives an informal derivation of the relationship between its area and circumference. As the sectors become smaller, their rearrangement (on right) more closely approximates a rectangle whose width is the area of the circle. The width is also half of the circumference (shown in black).

There are pedagogical choices to be made about how to treat the fundamental constant π. One option is to have students learn about the relationship between the area and circumference of a circle before introducing the name of the constant involved. Because a diagram of a circle of radius r is a scale drawing of the circle of radius 1 with scale factor r, students can deduce that the area of the circle of radius r is proportional to the square of the radius. A scale drawing argument can also be used to see that the circumference of a circle is proportional to its radius. Finally, a dissection argument suggests how the area and circumference of a circle are related. Putting these together: If A is the area of a circle of radius r and C is its circumference, then $A = kr^2$ and $C = 2kr$ where k is the area of a circle of radius 1. Students can be told that k is known as π.

In Grade 7, students build on earlier experiences with angle measurement (see the Grade 4 section of the Geometric Measurement Progression) to solve problems that involve *supplementary angles*, *complementary angles*, *vertical angles*, and *adjacent angles*. Vertical angles have the same number of degrees because they are both supplementary to the same angle. Keeping in mind that two geometric figures are "the same" in Grade 7 if one can be superimposed on the other, it follows that angles that are the same have the same number of degrees. Conversely, if two angles have the same measurement, then one can be superimposed on the other, so having the same number of degrees is a criterion for two angles to be the same. An angle is called a *right angle* if, after extending the rays of the angle to lines, it is the case that all the angles at the vertex are the same. In particular, the measurement of a right angle is 90°. In this situation, the intersecting lines are said to be *perpendicular*. Knowledge of angle measurements allows students to use algebra to determine missing information about particular geometric figures,[7.G.5] using algebra in the service of geometry, rather than the other way around.

[7.G.5]Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Grade 8

Rigid Motions and Congruence In Grade 7, two figures are considered the same if they "match up," that is, one can be superimposed on the other. In Grade 8, students connect this idea with the properties of translations, rotations and reflections.[8.G.A] They experimentally verify these properties[8.G.1] and gain experience with them by using transparencies. The paper below is fixed, and the transparency above is moved, illustrating the motion. A *translation* slides the transparency in a particular direction for a particular distance, keeping horizontal lines horizontal; a *rotation* rotates the transparency around a particular point, the *center of the rotation*, through a particular angle; and a *reflection* flips the transparency over a particular line, the *line of the reflection*. Reflections, rotations, and translations, and compositions of these, are called *rigid motions*.• Students manipulate these and observe they preserve the lengths of line segments and the measurements of angles. Terminology for transformations—for example *image*, *pre-image*, *preserve*—may be introduced in response to the need to describe the effects of rigid motions and other transformations.

Initially, students view rigid motions as operations on figures. Later, students come to understand that it is not the figure that is translated, rotated, or reflected, it is the plane that is moved, carrying the figure along with it. Students start thinking, not of moving one figure onto another, but of moving the plane so that the first figure lands on the second. The point of this change is that it becomes possible to describe the effect of these motions in terms of coordinates. Special cases are investigated in Grade 8, and the idea is fully developed in high school.

Two figures in the plane are said to be *congruent* if there is a sequence of rigid motions that takes one figure onto the other.[8.G.2] It should be noted that if we find a sequence of rigid motions taking figure A to figure B, then we can also find a sequence taking figure B to figure A. In high school mathematics the topic of congruence will be developed in a coherent, logical way, giving students the tools to investigate many geometric questions. In Grade 8, the treatment is informal, and students discover what they can about congruence through experimentation with actual motions.

As students perform transformations on the coordinate plane they discover the relationship between the coordinates of the image and the pre-image under rigid motions.[8.G.3] In Grade 8, the list of transformations for which this is feasible is quite short: translations, reflections in the axes, and rotations by 90° and 180°.

Dilations and Similarity In Grade 7, students study scale drawings to as a prelude to the transition from "same shape" to similarity in Grade 8. In Grade 8, change in scale becomes understood in terms of transformations that expand or contract the plane and the previous

[8.G.A] Understand congruence and similarity using physical models, transparencies, or geometry software.

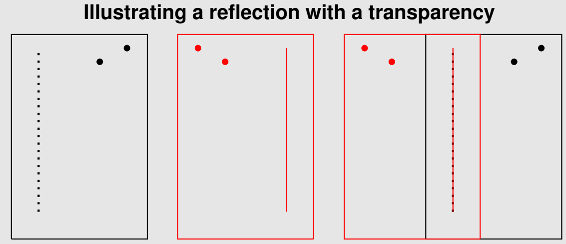
[8.G.1] Verify experimentally the properties of rotations, reflections, and translations:

a Lines are taken to lines, and line segments to line segments of the same length.

b Angles are taken to angles of the same measure.

c Parallel lines are taken to parallel lines.

Illustrating a reflection with a transparency



Trace the line of reflection (dotted) and the figure to be reflected (two dots) from a piece of paper (in black) onto a transparency (in red). Turn the transparency over and superimpose the red and dotted black reflection lines.

• Students should get a sense that rigid motions are special transformations. They should encounter and experience transformations which do not preserve lengths, do not preserve angles, or do not preserve either.

[8.G.2] Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

work with scale drawings flows naturally into describing dilations in terms of coordinates.[8.G.3]

Students observe the properties of dilations by experimenting with them, just as they did with rigid motions. They notice that shape is preserved under dilations, but that size is not preserved unless $r = 1$. This observation suggests that we make the idea of "same shape" can be made precise as similarity: Two figures are *similar* if there is a sequence of rigid motions and dilations that places one figure directly on top of the other.[8.G.4]

An important use of these properties is that of verifying that the slope of a (non-vertical) line can be determined by any two points on the line.[8.EE.6] See the Expressions and Equations Progression.

Parallel Lines, Transversals, and Triangles In Grade 8, students build on their experimentation with triangles in Grade 7 and start to make informal arguments about their properties.[8.G.5] They begin to see how rigid motions can play a role in such arguments.

For example, in Euclid's *Elements* two lines are defined to be parallel if they have no point of intersection. This definition requires imagining the two lines on the plane, which extend in two opposite directions, and checking to see that the two lines do not intersect anywhere along their infinite lengths. The *Elements* gives another way of thinking about what it means for two lines to be parallel: Given two lines L and L', draw a third line L'' (called a *transversal*) that intersects both. The lines L and L' are *parallel* if corresponding angles at their points of intersection with L'' are the same. This requires imagining only a finite section of the plane large enough to include segments from corresponding angles (as in the margin).

Students can use the properties of rotations and the figure in the margin to understand why Euclid's definition of parallel lines might be equivalent to the less intuitive characterization given in terms of corresponding angles.

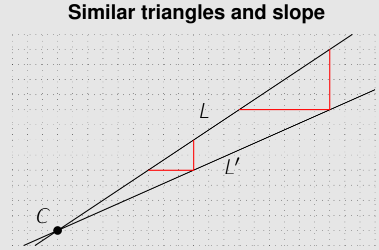
This discussion of lines and angles can continue to consider constraints on angles in triangles. First, given two (interior) angles in a triangle, their sum must be less than 180°. If the sum were equal to 180°, then the triangle would have two parallel sides, and thus no third vertex. From here students might get evidence about the sum of the three angles of a triangle by drawing their their favorite triangles, cutting out the angles, and putting their vertices together. This is an opportunity to distinguish between corrected and flawed reasoning (MP.3). If all 30 students in the class cut out the angles of their 30 triangles, and upon placing the vertices together get an almost straight angle, is this proof of the assertion that the sum of the angles of a triangle is a straight angle? Should they conclude instead that the sum is almost always a straight angle, but there may be exceptions? Most importantly, *why* is it true that the sum of the angle is 180°?[8.G.5]

[8.G.3] Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

[8.G.4] Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

[8.EE.6] Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

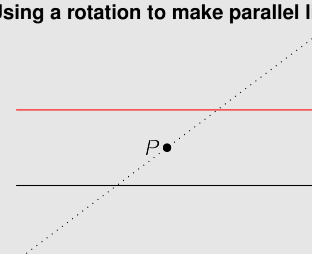
Similar triangles and slope



The line L' goes through the vertices of the right angles of the slope triangles for line L. If L and L' are parallel, there is a translation that maps one slope triangle to another. If L and L' are not parallel, they intersect in a point C (as shown above) and there is a dilation with center C that takes one triangle onto the other. In either case, the quotients of vertical and horizontal leg lengths for the two triangles are the same.

[8.G.5] Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

Using a rotation to make parallel lines



A 180° rotation about point P maps the line through P to itself. The image of the black line is the red line. Because a rigid motion takes an angle to an angle of the same measure, corresponding angles in this figure must be equal. The two horizontal lines cannot intersect; if they did, their images under the rotation would also intersect. (For example, if the lines intersected to the left of P in the pre-image, the image of their intersection would appear to the right of P.) If the two lines intersect in two distinct points, they must be the same line. Thus, the red and black lines are parallel according to Euclid's definition.

[8.G.5] Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

Understand and apply the Pythagorean Theorem. In Grade 7, while exploring the question, "What determines a unique triangle?," students might find that a right triangle is determined by the lengths of any two of its sides. In Grade 8, students might ask: "How do we find the length of the third side, knowing the lengths of two sides?"—beginning a series of investigations that leads naturally to the Pythagorean Theorem and its converse.

Students learn that there are lengths that cannot be represented by a rational number. For example, by looking at areas of figures in the coordinate plane, students discover that the hypotenuse of a triangle with legs of length 1 is an irrational number (see margin). Students can continue this line of reasoning to explain a dissection proof of the Pythagorean Theorem.[8.G.6] And here, it is essential to avoid the algebra involving the expansion of $(a+b)^2$, since that is not Grade 8 algebra. There are many proofs without words or symbols on the Internet. Not only is this visually more convincing, but it provides a purely geometric proof, consistent with the theme of Euclid's *Elements*.

This is an opportunity to discuss the meaning of *converse*, and the converse of the Pythagorean Theorem: a triangle with side lengths satisfying $c^2 = a^2 + b^2$ must be a right triangle with the right angle opposite the side of length c. Tradition has it that ancient Egyptian surveyors used the converse to construct right angles. They carried a loop of rope with 12 equally spaced knots. By pulling the rope taught, insisting that there be an angle at the fourth knot, and another at the seventh knot, they guaranteed that the angle at the fourth knot is a right angle: the triangle with side lengths 3,4,5 is a right triangle. This (with knots replaced by markings) is the method recommended by the United Nations Food and Agriculture Organization.[1]

An argument for the converse of the Pythagorean Theorem can be given. Recall the discussion of uniqueness in Grade 7: a triangle with given side lengths is unique. Suppose there is some triangle T with side lengths a, b, and c such that $c^2 = a^2 + b^2$. Must T be a right triangle? By the Pythagorean Theorem, there is a right triangle with the same side lengths that T has, namely the right triangle with legs a and b and hypotenuse c. Because a triangle with given side lengths is unique, T must be that right triangle. Note that this argument implicitly uses the SSS criterion for congruence.

In Grade 6, students calculate distances in the coordinate plane between points lying on the same horizontal or vertical line. In particular, they calculate the lengths of the vertical and horizontal legs of a slope triangle corresponding to two points in the coordinate plane. In Grade 7, they can use the Pythagorean Theorem to calculate the length of its hypotenuse, which is the distance between the two points.[8.G.8] Calculating this distance as an application of the Pythagorean Theorem before doing so in high school as an applica-

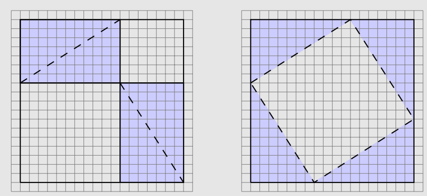
[1] See http://www.fao.org/docrep/R7021E/r7021e05.htm.

Squares in the coordinate plane

The square on the right has area 4. The square inside it (which is also the square on the left) must have area 2, because it is obtained by subtracting from the large square four triangles of area $\frac{1}{2}$. Because its area is 2, the side length of this square must be $\sqrt{2}$.

[8.G.6] Explain a proof of the Pythagorean Theorem and its converse.

Dissection that explains the Pythagorean Theorem



[8.G.8] Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

tion of the distance formula provides students an opportunity to look for and make use of structure in the coordinate plane (MP.7), and provides an opportunity for students to connect the distance formula to previous learning.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. In elementary grades, students became familiar with cubes, prisms, cones, and cylinders.[1.G.2] They calculated the volumes of right rectangular prisms—with whole-number edge lengths in Grade 5, and with fractional edge lengths in Grade 6. In Grade 7, they examined cross-sections of right rectangular prisms and pyramids, and calculated volumes of right prisms.[7.G.6] In Grade 8, students work with a wider variety of three-dimensional figures, including non-right figures.

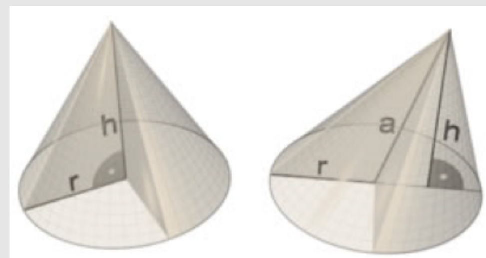
Students learn and use formulas for the volumes of cylinders, cones, and spheres.[8.G.9] Explanations for these formulas do not occur until high school.[G-GMD.1] However, Grade 8 students can look for structure in these formulas (MP.7). They know that the volume of a cube with sides of length s is s^3. A cube can be decomposed into three congruent pyramids, each of which has a square base, where the height is equal to the side length of the square. Each of these pyramids must have the volume $\frac{1}{3}s^3$, suggesting that the volume of a pyramid whose base has area b and whose height is h might be $\frac{1}{3}bh$. The volume formulas for cylinders and cones have an analogous relationship.

$$\begin{array}{rl} \text{cylinder} & bh = \pi r^2 h \\ \text{cone} & \dfrac{1}{3}bh = \dfrac{1}{3}\pi r^2 h \end{array}$$

[1.G.2]Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[2]

[7.G.6]Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Different cones with the same base

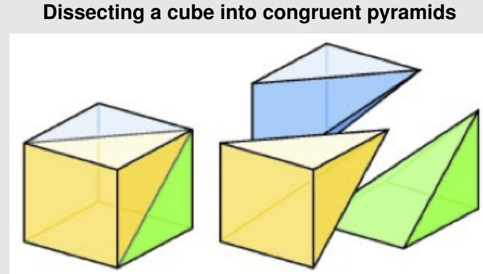


A *cone* is formed by drawing segments from a two-dimensional figure to a point that lies outside the plane of the figure; to be precise, it is the set of all line segments from the point to the figure. The two-dimensional figure is called the *base* of the cone, and the point, its *apex*. The two cones shown above have the same base (a circle) but different heights. The height of the cone on the left intersects the center of the base (thus, the cone is a right cone), the other height does not.

[8.G.9]Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

[G-GMD.1] Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

Dissecting a cube into congruent pyramids



High School

One of the problems encountered by learners in geometry is the formalism of prevailing presentations of the subject. The two basic concepts of congruence and similarity come across as either formal and abstract, or pleasant but irrelevant. In the axiomatic presentations, congruence and similarity are defined only for polygons, and as such they are divorced from the way these terms are used in the intuitive context. In the other extreme, congruence is "same size and same shape," and similarity is "same shape but not necessarily the same size," vague expressions that are often not connected with techniques for proving theorems such as the triangle congruence and similarity criteria (SAS, ASA, SSS, and AA).

The approach taken in the Standards is intended to avoid the pitfalls associated with both of these extremes. Instead of being vaguely defined or defined only for polygons, congruence and similarity are defined in terms of transformations. In particular, congruence is defined in terms of rigid motions—reflections, rotations, and translations. The potential benefit of this definition is that the abstract concept of congruence can then be grounded in kinetic and tactile experiences. This is why the Grade 8 geometry standards ask for the use of manipulatives, especially transparencies, to illustrate reflections, rotations, and translations, i.e., to illustrate congruence. In high school, students use the properties of reflections, rotations, and translations to *prove* the three congruence criteria. In this approach, proving theorems in geometry does not have to be an exercise in formalism and abstraction. Congruence is something students can relate to in a tactile manner just by moving a transparency over a piece of paper. Likewise, the learning of similarity can be grounded in tactile experiences.

There is also another advantage of this approach to congruence and similarity. Because most of the theorems in plane geometry before the introduction of similarity depend only on the three triangle congruence criteria, once these have been established, it is possible to transition into the traditional way of proving theorems at this point, without further use of basic rigid motions, if so desired. The use of dilations to treat similarity can likewise be limited to the initial stage if so desired.[3]

Congruence

G-CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

The different tools available to students for studying geometry in high school—straightedge and compass, transparencies or translucent paper, dynamic geometry software—lead to different insights and understandings, and can be used throughout for different purposes (MP5). Early experience with simple constructions, such as construction of a perpendicular to a line or of a line through a given

[3] See http://math.berkeley.edu/~wu/CCSS-Geometry.pdf

point parallel to a given line,[G-CO.12] can give a specificity to geometric concepts that can serve as a good basis for developing precise definitions and arguments.[G-CO.1] For example, the act of setting compass points to a given length and then drawing a circle with a given center makes concrete the formal definition of a circle. Two distinct lines are defined to be parallel if they do not intersect. In high school, students formalize an important understanding about parallel lines as the Parallel Postulate.•

Later in their studies, students can use dynamic geometry software to visualize theorems. The fact that the medians of a triangle always intersect in a point is more remarkable when the triangle is moving and changing shape. Geometric constructions and the tools for making them can be woven through a geometry course.

Experiment with transformations in the plane Students in high school start to formalize the intuitive geometric notions they developed in Grades 6–8.[G-CO.1] For example, in Grades 6–8 they worked with circles and became familiar with the idea that all the points on a circle are the same distance from the center. In high school, this idea underlies the formal definition of a circle: given a point O and a positive number r, a circle is the set of all points P in the plane such that $|OP| = r$. This definition will be important in proving theorems about circles, for example the theorem that all circles are similar.

Students also formalize the notion of a transformation as a function from the plane to itself.[G-CO.2] When the transformation is a rigid motion (a translation, rotation, or reflection) it is useful to represent it using transparencies because two copies of the plane are represented, one by the piece of paper and one by the transparency. These correspond to the domain and range of the transformation, and emphasize that the transformation acts on the entire plane, taking each point to another point. The fact that rigid motions preserve distance and angle is clearly represented because the transparency is not torn or distorted.

Constructing the results of transformations using a straightedge and compass can also bring out their functional aspect. For example, given a directed line segment AB and a point P, students can construct a line through P parallel to AB, then mark off the distance $|AB|$ along that line to construct the image of P under translation.

Transparencies are particularly useful for representing the symmetries of a geometric figure,[G-CO.3] because they keep the original and transposed figure on separate planes, something that can only be imagined when you are using geometry software.

Building on their hands-on work with rigid motions, students learn mathematical definitions of them (see margin).[G-CO.4] These definitions serve as the logical basis for all the theorems that students prove in geometry. Three basic properties of rigid motions are taken

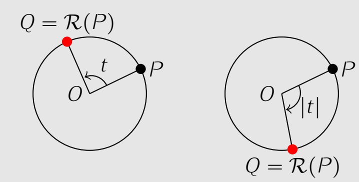
[G-CO.1] Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

• The Parallel Postulate

Given a line ℓ and a point P not on the line, there is exactly one line through P parallel to ℓ.

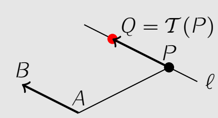
Students are not required to explain why this formulation of the Parallel Postulate (known as Playfair's Axiom) is equivalent to Euclid's Fifth Postulate, however, see the Grade 8 discussion of parallel lines for an illustration of this equivalence.

Definition of rotation



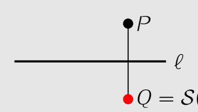
The rotation $\mathcal{R}$ around the point O through the angle t takes a point P to the point $Q = \mathcal{R}(P)$ as follows. If $P = O$, then $\mathcal{R}(O) = O$. If $P \neq O$ and $t \geqslant 0$, then Q is on the circle with center O and radius $|OP|$ so that $\angle POQ = t^\circ$ and Q is counterclockwise from P. If $t < 0$, we rotate clockwise by $|t|^\circ$.

Definition of translation



The translation $\mathcal{T}$ along the directed line segment $\vec{AB}$ takes the point P to the point $Q = \mathcal{T}(P)$ as follows. Draw the line ℓ passing through P and parallel to line through A and B. Then Q is the point on ℓ so that the direction from P to Q is the same as the direction from A to B and so that $|PQ| = |AB|$.

Definition of reflection



The reflection $\mathcal{S}$ across the line ℓ take each point on ℓ to itself, and takes any other point P to the point $Q = \mathcal{S}(P)$ which is such that ℓ is the perpendicular bisector of the segment PQ.

[G-CO.2] Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

[G-CO.3] Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

[G-CO.4] Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

as axiomatic, that is, as not needing proof. All rigid motions are assumed to

- map lines to lines, rays to rays, and segments to segments.
- preserve distance.
- preserve angle measure.

In Grade 8, students described sequences of rigid motions informally and in terms of coordinates. An important step forward in high school is to give precise descriptions of sequences of rigid motions that carry one figure onto another.[G-CO.5] Each rigid motion must be specified: For each rotation, a specific point and angle must be given; each translation is determined by a pair of points; and each reflection by a specific line, known as *the line of reflection*. These points, lines and angles must be described in terms of the two figures (see margin).

Understand congruence in terms of rigid motions Two figures are defined to be congruent if there is a sequence of rigid motions carrying one onto the other.[G-CO.6] It is important to be wary of circularity when using this definition to establish congruence. For example, you cannot assume that if two triangles have corresponding sides of equal length and corresponding angles of equal measure then they are congruent; this is something that must be proved using the definition of congruent, as shown in the margin.[G-CO.7]

Notice that the argument in the margin does not in fact use every equality of corresponding sides and angles. It only uses $|BC| = |QR|$, $m\angle ACB = m\angle PRQ$, and $|CA| = |RP|$ (along with the fact that rigid motions preserve all of these equalities). These equalities are indicated with matching hash marks in the figures. Thus, this argument it amounts to a proof of the SAS criterion for congruence. A variation of this argument can also be used to prove the ASA criterion. In that case one would drop the assumption that $|AC| = |PR|$ (the double hash marks) and add the assumption that $m\angle ABC = m\angle PQR$. Then one would argue at the conclusion that P coincides with the reflection $\mathcal{R}(\mathcal{T}(A))$ because line QP coincides with the reflection of line $Q\mathcal{R}(\mathcal{T}(A))$, and therefore its intersection with line RP must coincide with the reflection of $\mathcal{R}(\mathcal{T}(A))$. The proof of the SSS congruence criterion[G-CO.8] is a little more involved.[4]

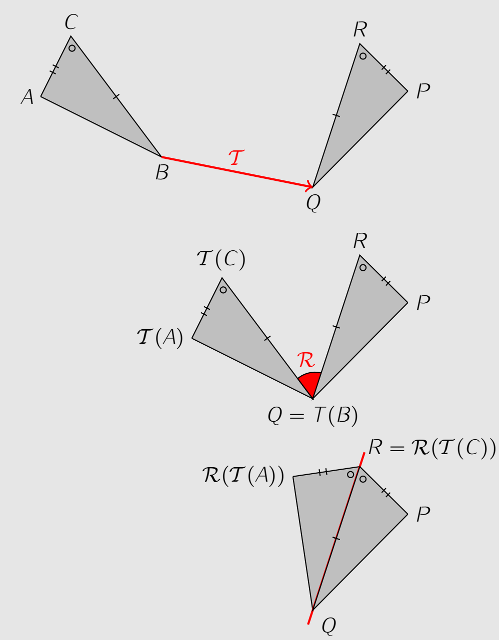
Prove geometric theorems Once the triangle congruence criteria are established using the transformation definition of congruence in terms of rigid motions, a geometry course can proceed to prove other theorems in the traditional way. Alternatively, a course could take advantage of the transformation definition of congruence to

[G-CO.5] Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

[G-CO.6] Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

[G-CO.5] Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Proving congruence



Suppose that the corresponding sides and corresponding angles of $\triangle ABC$ and $\triangle PQR$ are equal. First translate $\triangle ABC$ along the line segment BQ, so that $\mathcal{T}(B) = Q$. Then rotate clockwise about Q through the angle $\angle\mathcal{T}(C)QR$. Because translation and rotation preserve distance, we have $R = \mathcal{R}(\mathcal{T}(C))$. Now reflect across the line through R and Q. Because the rigid motions preserve angles, the line through R and P coincides with the reflection of the line through R and $\mathcal{R}(\mathcal{T}(A))$, and because they preserve distance the point P coincides with the reflection of $\mathcal{R}(\mathcal{T}(A))$. Now all three points of the triangle coincide, so we have produced a sequence of rigid motions that maps $\triangle ABC$ onto $\triangle PQR$, and they are therefore congruent.

[G-CO.8] Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

[4] See, for example, page 148 of Wu, `https://math.berkeley.edu/~wu/Progressions_Geometry.pdf`.

give particularly simple proofs of some theorems. For example, the theorem that the base angles of an isosceles triangle are congruent can be proved by reflecting the triangle about the bisector of the angle between the two congruent sides.

Fertile territory for exercising students' reasoning abilities is the proof that various geometric constructions are valid, that is, they construct the figures that they are intended to construct.[G-CO.13]

Similarity

Understand similarity in terms of similarity transformations The concept of similarity builds on the concept of congruence, and so is introduced after it, following a progression like that for congruence. As with rigid motions, student develop the notion of dilation they developed in Grade 8 into a formal definition of a dilation as a function on the plane. Two figures are defined to be similar if there is a sequence of rigid motions and dilations which takes one to the other. Equivalently, and more conveniently in many arguments, we can say that two figures are similar if one is congruent to a dilation of the other.

As with rigid motions, students get hands-on experience with dilations in Grade 8, using graph paper or dynamic geometry software. In high school, they observe basic properties of dilations. For example, they observe experimentally that a dilation takes a line to to another line which is parallel to the first, or identical to it if the line is through the center of dilation.[G-SRT.1a] This important fact is used repeatedly in later work.

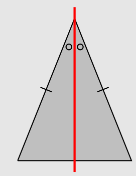
The traditional notion of similarity applies only to polygons. Two such figures are said to be similar if corresponding angles are congruent and corresponding lengths are related by a constant scale factor. If similarity is defined in terms of transformations, then this understanding is a consequence of the definition rather than being a definition itself.[G-SRT.2]

For example, using the fact that a dilation takes a line to a parallel line, and facts about transversals of pairs of parallel lines, students can show that a dilation preserves angles, that is, the image of an angle is congruent to the angle itself (see margin).

They also observe that under dilation the length of any line segment—not only segments with an endpoint at the center—is scaled by the scale factor of the dilation.[G-SRT.1b] A complete proof of this fact is beyond the scope of a high school geometry course, but a proof in specific cases is suitable for an investigation by STEM-intending students. The simplest case is where the scale factor is 2, which can be proven by a congruence argument.

Conversely, students can see that two figures which are similar according to the traditional notion are also similar according to the transformation definition by deriving the AA criterion for similarity of triangles (see margin on next page).[G-SRT.3]

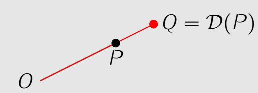
The base angles of an isosceles triangle are congruent



Because reflections preserve angle and length, the two congruent sides are taken to each other, and therefore the reflection takes the triangle to itself. This means it maps the base angles onto each other, and so they must be congruent.

G-CO.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Definition of dilation



The dilation $\mathcal{D}$ with center O and positive scale factor r leaves O unchanged and takes every point P to the point $Q = \mathcal{D}(P)$ on the ray OP whose distance from O is $r|OP|$.

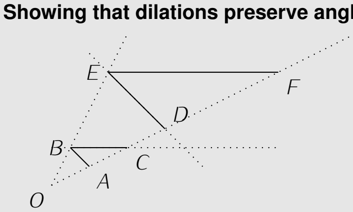
G-SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

a A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

G-SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Showing that dilations preserve angles



The dilation with center O takes $\angle ABC$ to $\angle DEF$. The line EF is parallel to the line BC. Extending segment ED to a transversal of these two parallel lines and using the fact that alternate interior angles are congruent, we see that $\angle ABC$ is congruent to $\angle DEF$.

G-SRT.3 Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

An advantage of the transformational approach to similarity is that it allows for a notion of similarity that extends to all figures rather than being restricted to figures composed of line segments. For example, consider a dilation of a circle whose center is the center of the dilation. Every point on the circle moves the same distance away, because they were originally all at the same distance from the center. Thus the new figure is also a circle. This reasoning can also be reversed to show that any two circles with the same center are similar. Furthermore, because any circle can be translated so that its center coincides with the center of any other circle, we can see that all circles are similar.[G-C.1]

Prove theorems involving similarity Students may have already seen a dissection proof of the Pythagorean Theorem which depends on congruence criteria for triangles. Now they can see a proof that uses the concept of similarity. This proof is an example of seeing structure (MP.7), because it requires constructing an auxiliary line, the altitude from the right angle to the hypotenuse, which reveals a decomposition of the triangle into two smaller similar triangles. The proof combines geometric insight and algebraic manipulation. [G-SRT.4]

Define trigonometric ratios and solve problems involving right triangles Because all right triangles have a common angle, the right angle, the AA criterion becomes, in the case of right triangles, an "A criterion"; that is, two right triangles are similar if they have an acute angle in common. This observation is the key to defining a trigonometric ratio• for a single acute angle.[G-SRT.6]

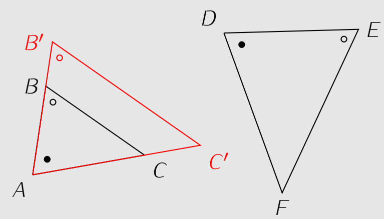
Analytic Geometry

The introduction of coordinates into geometry connects geometry and algebra, allowing algebraic proofs of geometric theorems. This area of geometry is called *analytic geometry*.

The coordinate plane consists of a grid of horizontal and vertical lines; each point in the plane is labeled by its displacement (positive or negative) from two reference lines, one horizontal and one vertical, called coordinate axes. Note that it is the displacement *from* the axes, rather than the displacement *along* these axes, that is usually the most useful concept in proving theorems. This point is sometimes obscured for students by the mathematical convention of putting a scale on the axes themselves and labeling them x-axis and y-axis. A different convention, of putting the scale within a quadrant (e.g., as can be done with dynamic geometry software), is sometimes used, and might be more useful pedagogically (see margin on next page).

From their work in Grade 8, students are familiar with the idea that two points (x_1, y_1) and (x_2, y_2) in the coordinate plane determine a right triangle whose hypotenuse is the line segment between the two points and whose legs are parallel to the axes.• Two im-

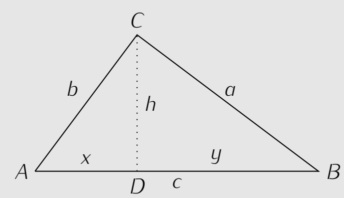
Proof of the AA criterion using similarity



Given $\triangle ABC$ and $\triangle DEF$ with $m\angle A = m\angle D$ and $m\angle B = m\angle E$, perform a dilation on $\triangle ABC$ with center at A so that $|AB'| = |DF|$. Because dilations preserve angles, $m\angle B' = m\angle E$, and so $\triangle AB'C'$ is congruent to $\triangle DEF$ by the ASA criterion. Since $\triangle AB'C'$ is a dilation of $\triangle ABC$, this means that $\triangle ABC$ is similar to $\triangle DEF$.

G-C.1 Prove that all circles are similar.

The Pythagorean Theorem using similarity



Given a right triangle ABC with right angle at C, drop an altitude from C to AB to decompose the triangle into two smaller triangles. Using the facts that the sum of the angles at C is $90°$ and the sum of the angles in each triangle is $180°$, we see that $\angle DAC$ is congruent to $\angle DBC$. Also, all three triangles have a right angle. So, by the AA criterion for similarity, $\triangle ACD$ and $\triangle CBD$ are similar to $\triangle ABC$, so

$$\frac{a}{c} = \frac{y}{a} \quad \text{and} \quad \frac{b}{c} = \frac{x}{b}$$

and therefore

$$a^2 + b^2 = c(x + y) = c^2.$$

G-SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

• Traditionally, trigonometry concerns "ratios." Note, however, that according to the usage of the Ratio and Proportional Reasoning Progression, that these would be called the "value of the ratio." In high school, students' understanding of ratio may now be sophisticated enough to allow "ratio" to be used for "value of the ratio" in the traditional manner. Likewise, angles are carefully distinguished from their measurements in Grades 4 and 5. In high school, students' understanding of angle measure may now allow angles to be referred to by their measurements.

G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

• The triangle is degenerate, collapsing to a line, if the line is horizontal or vertical.

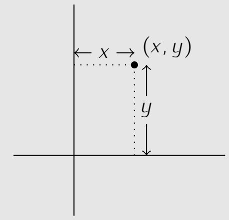
portant geometric facts about these triangles lead to foundational formulas in analytic geometry.

First, for all the pairs of distinct points on a given line, the corresponding triangles are similar. This can be shown using the AA criterion for similarity. Because the horizontal (or vertical) grid lines are all parallel to each other, and the line is transversal to those parallel lines, the ratio of the vertical side to the horizontal side does not depend on which two points are chosen, and so is a characteristic of the line itself, called its slope, m. The algebraic manifestation of this is the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

The relationship between the slopes of parallel and perpendicular lines is a nice example of the interplay between geometry and algebra.[G-GPE.5]

Displaying the coordinates of a point



A standard convention is the use of x to represent the horizontal displacement of a point from the vertical axis and y to represent its vertical displacement from the horizontal axis. Showing these displacements within a quadrant rather than on the x- and y-axes emphasizes the fact that they can be viewed as distances from the axes as well as distances along the axes.

G-GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Second, the Pythagorean Theorem applies: the length of the hypotenuse is the distance between the two points, and the lengths of the legs can be calculated as differences between the coordinates. The algebraic manifestation of the Pythagorean Theorem is the formula for the distance, d, between the two points:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Students can use the distance formula to prove simple facts about configurations of points in the plane.[G-GPE.4]

G-GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $(1, \sqrt{3})$ *lies on the circle centered at the origin and containing the point* $(0, 2)$.

However, the power of analytic geometry to reduce geometric relationships to algebraic ones is a danger in teaching it because students can lose sight of the geometric meaning of the formulas. Thus the equation for a circle with center (a, b) and radius r,

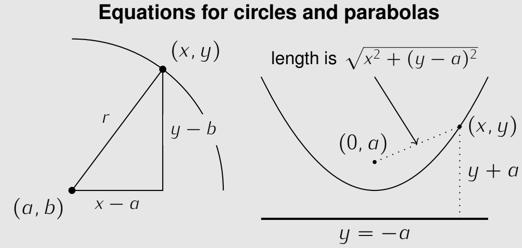
$$(x - a)^2 + (y - b)^2 = r^2$$

can become disconnected from the Pythagorean theorem, even though it is nothing more than a direct statement of that theorem for any right triangle with radius of the circle as its hypotenuse.[G-GPE.1]

G-GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

As another example, students sometimes get the the impression that the word "parabola" is the name for the graph of a quadratic function, whereas a parabola is a geometric object with a geometric definition. It is a beautiful and simple exercise in the interplay between geometry and algebra to derive the equation from this definition.[G-GPE.2]

Equations for circles and parabolas



Applying the Pythagorean Theorem to the triangle on the left yields the equation for a circle, $(x - a)^2 + (y - b)^2 = r^2$.

On the right is parabola, defined geometrically by the condition that a point on the parabola is equidistant from the focus (at $(0, a)$) and the directrix (the line $y = -a$). Setting these two distances equal and squaring both sides yields $x^2 + (y - a)^2 = (y + a)^2$, which reduces to the familiar equation $y = (1/4a)x^2$.

G-GPE.2 Derive the equation of a parabola given a focus and directrix.

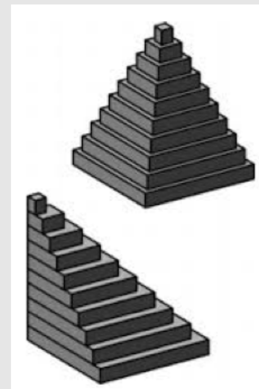
Geometric Measurement

In Grade 8, students learned the formulas for the volumes of cones, cylinders, and spheres. In high school, they give informal justifications of these formulas.[G-GMD.1] The cube dissection argument on page 12 verifies the formula for the volume of a specific pyramid with a square base. In high school, students view the pyramid as a stack of layers, and, using CavalieriâĂŹs Principle, see that shifting its layers does not change its volume. Furthermore, stretching the height of the pyramid by a given scale factor thickens each layer by the scale factor, and so multiplies its volume by that factor. Using such arguments, students can derive the formula for the volume of any pyramid with a square base. A further exploration using dissection arguments, transformations of layers, and informal limit arguments, can lead to the general formula for the volume of a cone.

A more complex argument in terms of layers derives the formula for the volume of a sphere from the formula for the volume of a cone.[5]

G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

Shifted layers of a pyramid



Geometry and Modeling

Any mathematical object that represents a situation from outside mathematics and can be used to solve a problem about that situation is a mathematical model. Modeling often involves making simplifying assumptions that ignore some features of the situation being modeled. If a population grows by approximately the same percentage each year, sometimes a bit above, sometimes a bit below, students might choose to fit an exponential function to the data and use it to make predictions. In geometry, in order to study how the illuminated percentage of the moon's surface varies during a month, students might represent the moon as a rotating sphere, half black and half white.[G-MG.1]

Geometric modeling can be used in *Fermi problems*, problems which ask for rough estimates of quantities. Such problems often involved estimates of densities, as in the example in the margin.[G-MG.2]

Of all the subjects students learn in geometry, trigonometry may have the greatest application in college and career. Students in high school should see authentic applications of trigonometry to many different contexts (see next page).[G-SRT.8]

G-MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

G-MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

A Fermi Problem: How Many Leaves on a Tree?

Amy and Greg are raking up leaves from a large maple tree in their yard and Amy remarks "I'll bet this tree has a million leaves." Greg is skeptical. Amy suggests the following method to check whether or not this is possible:

- Find a small maple tree and estimate how many leaves it has.
- Use that number to figure out how many leaves the big maple tree has.

1. Describe the assumptions and calculations needed to carry out Amy's strategy.
2. Amy and Greg estimate that their maple tree is about 35 feet tall. They find a 5-foot-tall maple tree and estimate that it has about 400 leaves. Use the calculations that you described to estimate the number of leaves on Amy and Greg's tree.

(adapted from Illustrative Mathematics, `illustrativemathematics.org/content-standards/tasks/1137`)

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

[5] See, for example, `http://www.matematicasvisuales.com/english/html/history/cavalieri/cavalierisphere.html`.

Modeling with trigonometry

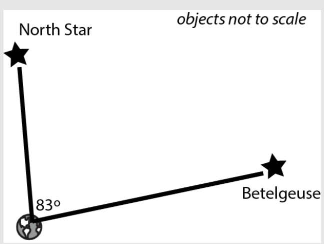
Vector graphic
A digital artist using a computer drawing program clicked on a diagonal line segment and saw that it measured 216 units horizontally and 45 units vertically. How many units long was the line segment? If the artist wants to rotate the line segment to be vertical, what angle of rotation could be used? Give your answers to the nearest hundredth.

Answers. 220.64 units long; 11.77° clockwise or 78.23° counter-clockwise.

Flight of the bumblebee
A bumblebee sitting on a tulip wanted to fly over to a daffodil located 100 meters due west. The bumblebee did fly in a straight line, but it mistakenly flew in a direction 5 degrees south of west. The bumblebee passed to the south of the daffodil—how far south?

Answer. 8.75 meters or about 9 meters.

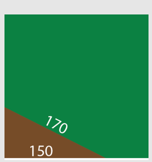
Star distance
Viewed from Earth, the North Star and Betelgeuse are separated by an angle of about 83 degrees. If the North Star is about 430 light years from Earth, and Betelgeuse is about 640 light years from Earth, then about how far apart are the North Star and Betelgeuse (measured in light years)?



Answer. About 726 light years.

Comment. This could be solved using a G-SRT.10(+) strategy (Law of Cosines), or via G-SRT.8 by dropping a perpendicular to make two right triangles.

Crop Loss
One corner of a soybean field wasn't irrigated, and no soybeans could be harvested from that part of the field. How much money was lost if soybeans sold for $12 a bushel that year and an acre of irrigated land yields 54 bushels of soybeans? Note, 1 acre is approximately 4000 square meters.



Answer: $972 or approximately $1000.

See `http://tinyurl.com/mtcn6zq`.