Common Core State Standards for Mathematics

Flip Book

Grade 3

Updated Fall, 2014

About the Flip Books

The development of the "flip books" is in response to the adoption of the Common Core State Standards by the state of Kansas in 2010. Teachers who were beginning the transition to the new Kansas Standards—Kansas College and Career Ready Standards (KCCRS) needed a reliable starting place that contained information and examples related to the new standards.

This project attempts to pull together, in one document some of the most valuable resources that help develop the intent, the understanding and the implementation of the KCCRS. The intent of these documents is to provide a starting point for teachers and administrators to begin unraveling the standard and is by no means the only necessary or complete resource that supports implementation of KCCRS.

This project began in the summer 2012 with the work of Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books". The "flip books" are based on a model that Kansas had for earlier standards however, this edition is far more comprehensive than those in the past. The current editions incorporate the resources from: other state departments of education, documents such as the content progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. The current product was a compilation of work from the project developers in conjunction with many mathematics educators from around the state. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KATM website at www.katm.org and will continue to undergo changes periodically. When significant changes/additions are implemented the necessary modification will be posted and dated.

The initial development of the current update to the "flip books" was driven by the need expressed by teachers of mathematics in Kansas and with the financial support from Kansas Department of Education and encouragement from the Kansas Association of Teachers of Mathematics. These "flip books" have become an ongoing resource that will continue to evolve as more is learned about high quality instruction for the KCCRS for mathematics.

For questions or comments about the flipbooks please contact Melisa at melisa@ksu.edu.

Planning Advice--Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptually understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

As the Kansas College and Career Ready Standards (KCCRS) are carefully examined, there is a realization that with time constraints of the classroom, not all of the standards can be done equally well and at the level to adequately address the standards. As a result, priorities need to be set for planning, instruction and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "While the remaining content is limited in scope." 4) a "lower" priority does not imply exclusion of content but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path, if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In planning for instruction "grain size" is important. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Daro (Teaching Chapters, Not Lessons—Grain Size of Mathematics), strands are too vague and too large a grain size, while lessons are too small a grain size. About 8 to 12 units or chapters produce about the right "grain size". In the planning process staff should attend to the <u>clusters</u>, and think of the standards as the ingredients of cluster, while understanding that coherence exists at the cluster level across grades.

A caution--Grain size is important but can result in conversations that do not advance the intent of this structure. Extended discussions that argue 2 days instead of 3 days on a topic because it is a lower priority detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, lenses focused on lessons can also provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics that follows presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they demine distribution of time for both planning and instruction, helping to assure that students really understand before moving on. Each cluster has been given a priority level. As professional staffs begin planning, developing and writing units as Daro suggests, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level by Zimba. The three levels are referred to as:—**Focus**, **Additional** and **Sample**. Furthermore, Zimba suggests that about 70% of instruction should relate to the Focus clusters. In planning, the lower two priorities (Additional and Sample) can work together by supporting the Focus priorities. The advanced work in the high school standards is often found in "Additional and Sample clusters". Students who intend to pursue STEM careers or Advance Placement courses should master the material marked with "+" within the standards. These standards fall outside of priority recommendations.

Recommendations for using cluster level priorities

Appropriate Use:
- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through: sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possibility quality; the additional work of the grade should indeed support the Focus priorities and not detract from it.
- Set priorities for other implementation efforts taking the emphasis into account such as: staff development; new curriculum development; revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:
- Neglecting any of the material in the standards rather than connecting the Additional and Sample clusters to the other work of the grade
- Sorting clusters from Focus to Additional to Sample and then teaching the clusters in order. To do so would remove the coherence of mathematical ideas and miss opportunities to enhance the focus work of the grade with additional clusters.
- Using the clusters' headings as a replacement for the actual standards. All features of the standards matter— from the practices to surrounding text including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise and coherence of the standards (grain size).

Each cluster, at a grade level, and, each domain at the high school, identifies five or fewer standards for in-depth instruction called Depth Opportunities (Zimba, 2011). Depth Opportunities (DO) is a qualitative recommendation about allocating time and effort within the highest priority clusters --the **Focus** level. Examining the Depth Opportunities by standard reflects that some are beginnings, some are critical moments or some are endings in the progressions. The DO's provide a prioritization for handling the uneven grain size of the content standards. Most of the DO's are not small content elements, but, rather focus on a big important idea that students need to develop.

DO's can be likened to the Priorities in that they are meant to have relevance for instruction, assessment and professional development. In planning instruction related to DO's, teachers need to intensify the mode of engagement by emphasizing: **tight focus**, **rigorous reasoning and discussion** and **extended class time devoted to practice and reflection** and have **high expectation for mastery**. (See Table 7 Appendix, Depth of Knowledge (DOK)
In this document, Depth Opportunities are highlighted pink in the Standards section. For example:

5.NBT.6 *Find whole number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays and/or area models.*

Depth Opportunities can provide guidance for examining materials for purchase, assist in professional dialogue of how best to develop the DO's in instruction and create opportunities for teachers to develop high quality methods of formative assessment.

Standards for Mathematical Practice in Grade 3

The Common Core State Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students Grades K-12. Below are a few examples of how these Practices may be integrated into tasks that Grade 2 students complete.

Practice	Explanation and Example
1) Make Sense and Persevere in Solving Problems.	Mathematically proficient students in Grade 3 examine problems, can make sense of the meaning of the task, and find an entry point or a way to start the task. Grade 3 students also develop a foundation for problem solving strategies and become independently proficient on using those strategies to solve new tasks. They might use concrete objects or pictures to show the actions of a problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. They are expected to persevere while solving tasks; that is, if students reach a point in which they are stuck, they can reexamine the task in a different way and continue to solve the task. Students in Grade 3 complete a task by asking themselves the question, "Does my answer make sense?" Example: to solve a problem involving multi-digit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.
2) Reason abstractly and quantitatively.	Mathematically proficient students in Grade 3 recognize that a number represents a specific quantity. They connect the quantity to written symbols and create a logical representation of the problem at hand, considering both the appropriate units involved and the meaning of the quantities. This involved two processes: decontextualizing and contextualizing. In Grade 3, students represent situations by decontextualizing tasks into numbers and symbols. For example, to find the area of the floor of a rectangular room that measures 10 ft by 12 ft, a student might represent the problem as an equation, solve it mentally, and record the problem and solution as 10 x 12 = 120. She has decontextualized the problem. When she states at the end that the area of the room is 120 square feet, she has contextualized the answer i order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (SMP 4).
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students in Grade 3 accurately use definitions and previously established solutions to construct viable arguments about mathematics. Grade 3 students might construct arguments using concrete referents such as objects, pictures, and drawings. They refine their mathematical communication skills as they participate in mathematical discussions involving questions like "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking. For example, when comparing the unit fractions 1/3 and 1/5, students may generate their own representation of both fractions and then critique each other's reasoning in class, as they connect their arguments to the representations that they created. Students in Grade 3 present their arguments in the form of representations, actions on those representations, and explanations in words (oral and written).

4) Model with mathematics.	Mathematically proficient students in Grade 3 experiment with representing problem situations in multiple ways, including numbers, words (mathematical language), drawing pictures, using objects, acting out, making a chart, list, or graph, creating equations, etc. They model real-life mathematical situations with a number sentence or an equation, and check to make sure that their equation accurately matches the problem context. Students should have ample opportunities to connect the different representations and explain the connections. Grade 3 students should evaluate their results in the context of the situation and reflect on whether the results make sense.
5) Use appropriate tools strategically.	Mathematically proficient students in Grade 3 consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be helpful. The tools that students in Grade 3 might use physical objects (place value (base ten) blocks, hundreds charts, number lines, tape diagrams, fraction bars, arrays, tables, graphs, and concrete geometric shapes (e.g. pattern blocks, 3-d solids) paper and pencil, rulers and other measuring tools, grid paper, virtual manipulatives, and concrete geometric shapes (e.g., pattern blocks, 3-d solids), etc. Students should also have experiences with educational technologies, such as calculators and virtual manipulatives that support conceptual understanding and higher-order thinking skills. During classroom instruction, students should have access to various mathematical tools as well as paper, and determine which tools are the most appropriate to use. For example, when comparing 4/6 and 1/2, students can use benchmark fractions and the number line and explain that 4/6 would be placed to the right of ½ because it is a "little bit more than ½." When students model situations with mathematics, they are choosing tools appropriately (SMP 5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (SMP2)
6) Attend to precision.	Mathematically proficient students in Grade 3 are precise in their communication, calculations, and measurements. In all mathematical tasks, they communicate clearly, using grade-level appropriate vocabulary accurately as well as giving precise explanations and reasoning regarding their process of finding solutions. For example, while measuring objects iteratively (repetitively), students check to make sure that there are no gaps or overlaps. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation. During tasks involving number sense, students check their work to ensure the accuracy and reasonableness of solutions.
7) Look for and make use of structure.	Mathematically proficient students in Grade 3 carefully look for patterns and structures in the number system and other areas of mathematics. Grade 3 students use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (SMP 8). For example, when Grade 3 students calculate 16 X 9, they might apply the structure of place value and the distributive property to find the product: 16 X 9 = (10 + 6) X 9 = (10 X 9) + (6 X 9).
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students in Grade 3 notice repetitive actions in computation and look for more shortcut methods. For example, students may use the distributive property as a strategy for using products they know to solve products that they don't know. For example, if students are asked to find the product of 7 X 8, they might decompose 7 into 5 and 2 and then multiply 5 X 8 and 2 X 8 to arrive at 40 + 16 or 56. Mathematically proficient 3rd graders formulate conjectures about what they notice. In addition, third graders continually evaluate their work by asking themselves, "Does this make sense?"

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
1. **Make sense of problems and persevere in solving them.** • Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem. • Plan a solution pathway instead of jumping to a solution. • Can monitor their progress and change the approach if necessary. • See relationships between various representations. • Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another. • Can understand various approaches to solutions. • Continually ask themselves; "Does this make sense?"	• How would you describe the problem in your own words? • How would you describe what you are trying to find? • What do you notice about? • What information is given in the problem? • Describe the relationship between the quantities. • Describe what you have already tried. • What might you change? • Talk me through the steps you've used to this point. • What steps in the process are you most confident about? • What are some other strategies you might try? • What are some other problems that are similar to this one? • How might you use one of your previous problems to help you begin? • How else might you organize, represent, and show?
2. **Reason abstractly and quantitatively.** • Make sense of quantities and their relationships. • Are able to decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships. • Understand the meaning of quantities and are flexible in the use of operations and their properties. • Create a logical representation of the problem. • Attends to the meaning of quantities, not just how to compute them.	• What do the numbers used in the problem represent? • What is the relationship of the quantities? • How is _____ related to _____? • What is the relationship between _____ and _____? • What does _____ mean to you? (e.g. symbol, quantity, diagram) • What properties might we use to find a solution? • How did you decide in this task that you needed to use? • Could we have used another operation or property to solve this task? Why or why not?
3. **Construct viable arguments and critique the reasoning of others.** • Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments. • Justify conclusions with mathematical ideas. • Listen to the arguments of others and ask useful questions to determine if an argument makes sense. • Ask clarifying questions or suggest ideas to improve/revise the argument. • Compare two arguments and determine correct or flawed logic.	• What mathematical evidence would support your solution? How can we be sure that _____? / How could you prove that _____? Will it still work if _____? • What were you considering when _____? • How did you decide to try that strategy? • How did you test whether your approach worked? • How did you decide what the problem was asking you to find? (What was unknown?) • Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not? • What is the same and what is different about _____? • How could you demonstrate a counter-example?
4. **Model with mathematics.** • Understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize). • Apply the math they know to solve problems in everyday life. • Are able to simplify a complex problem and identify important quantities to look at relationships. • Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation. • Reflect on whether the results make sense, possibly improving or revising the model. • Ask themselves, "How can I represent this mathematically?"	• What number model could you construct to represent the problem? • What are some ways to represent the quantities? • What's an equation or expression that matches the diagram, number line, chart, table? • Where did you see one of the quantities in the task in your equation or expression? • Would it help to create a diagram, graph, table? • What are some ways to visually represent? • What formula might apply in this situation?

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
5. Use appropriate tools strategically. • Use available tools recognizing the strengths and limitations of each. • Use estimation and other mathematical knowledge to detect possible errors. • Identify relevant external mathematical resources to pose and solve problems. • Use technological tools to deepen their understanding of mathematics.	• What mathematical tools could we use to visualize and represent the situation? • What information do you have? • What do you know that is not stated in the problem? • What approach are you considering trying first? • What estimate did you make for the solution? • In this situation would it be helpful to use: a graph, number line, ruler, diagram, calculator, manipulative? • Why was it helpful to use._____? • What can using a _____ show us, that _____may not? • In what situations might it be more informative or helpful to use._____?
6. Attend to precision. • Communicate precisely with others and try to use clear mathematical language when discussing their reasoning. • Understand meanings of symbols used in mathematics and can label quantities appropriately. • Express numerical answers with a degree of precision appropriate for the problem context. • Calculate efficiently and accurately.	• What mathematical terms apply in this situation? • How did you know your solution was reasonable? • Explain how you might show that your solution answers the problem. • Is there a more efficient strategy? • How are you showing the meaning of the quantities? • What symbols or mathematical notations are important in this problem? • What mathematical language, definitions, properties can you use to explain._____? • How could you test your solution to see if it answers the problem?
7. Look for and make use of structure. • Apply general mathematical rules to specific situations. • Look for the overall structure and patterns in mathematics. • See complicated things as single objects or as being composed of several objects.	• What observations do you make about._____? • What do you notice when._____? • What parts of the problem might you eliminate, simplify? • What patterns do you find in._____? • How do you know if something is a pattern? • What ideas that we have learned before were useful in solving this problem? • What are some other problems that are similar to this one? • How does this relate to._____? • In what ways does this problem connect to other mathematical concepts?
8. Look for and express regularity in repeated reasoning. • See repeated calculations and look for generalizations and shortcuts. • See the overall process of the problem and still attend to the details. • Understand the broader application of patterns and see the structure in similar situations. • Continually evaluate the reasonableness of their intermediate results.	• Will the same strategy work in other situations? • Is this always true, sometimes true or never true? • How would we prove that._____? • What do you notice about._____? • What is happening in this situation? • What would happen if._____? • What Is there a mathematical rule for._____? • What predictions or generalizations can this pattern support? • What mathematical consistencies do you notice?

Critical Areas for Mathematics in 3rd Grade

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

1. Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.
   (OA.1, OA.2, OA.3, OA.4, OA.5, OA.6, OA.7, OA.9)

2. Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, $1/2$ of the paint in a small bucket could be less paint than $1/3$ of the paint in a larger bucket, but $1/3$ of a ribbon is longer than $1/5$ of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.
   (NF.1, NF.2, NF.3)

3. Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.
   (MD.5, MD.6, MD.7)

4. Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.
   (NF.1,G.1,G.2)

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Operations and Algebraic Thinking (OA)
- Represents and solves problems involving multiplication and division
  OA.1   OA.2   OA.3   OA.4
- Understand properties of multiplication and the relationship between multiplication and division
  OA.5   OA.6
- Multiply and divide within 100
  OA.7
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.
  OA.8   OA.9

Number and Operations in Base Ten (NBT)
- Use place value understanding and properties of operations to perform multi-digit arithmetic.
  NBT.1   NBT.2   NBT.3

Number and Operations – Fractions (NF)
- Develop understanding of fractions as numbers.
  NF.1   NF.2   NF.3

Measurement and Data (MD)
- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
  MD.1   MD.2
- Represent and interpret data.
  MD.3   MD.4
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
  MD.5   MD.6   MD.7
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
  MD.8

Geometry (GE)
- Reason with shapes and their attributes
  G.1   G.2

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Operations and Algebraic Thinking (OA)
Cluster: Represents and solves problems involving multiplication and division.

Standard: Grade 3.OA.1
Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7   (OA)

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: (3.OA.1-4)
This cluster is connected to:
- Third Grade Critical Area of Focus #1**: Developing understanding of multiplication and division and strategies for multiplication and division within 100.**
- Connect this domain with understanding properties of multiplication and the relationship between multiplication and division. (Grade 3 OA 5 – 6)
- The use of a symbol for an unknown is foundational for letter variables in Grade 4 when representing problems using equations with a letter standing for the unknown quantity (Grade 4 OA 2 and OA 3).

Explanation and Examples:
The standard interprets products of whole numbers. Students need to recognize multiplication as a means of determining the total number of objects when there are a specific number of groups with the same number of objects in each group. Multiplication requires students to think in terms of *groups of things* rather than *individual things*. At this level, Multiplication is seen as "groups of" and problems such as 5 x 7 refer to 5 groups of 7.

It is important for teachers to understand there are several ways in which think of multiplication:
- Multiplication is often thought of as repeated addition of equal groups. While this definition works for some sets of numbers, it is not particularly intuitive or meaningful when we think of multiplying 3 by $\frac{1}{2}$, for example, or 5 by -2. In such cases, it may be helpful to widen the idea of grouping to include evaluation of part of a group. This concept is related to **partitioning** (which, in turn, is related to division).

  Ex: Three groups of five students can be read as 3 • 5, or 15 students, while half a group of 10 stars can be represented as $\frac{1}{2}$ · 10, or 5 stars. These are examples of partitioning; each one of the three groups of five is part of the group of 15, and the group of 5 stars is part of the group of 10.

- A second concept of multiplication is that of **rate or price**. Ex: If a car travels four hours at 50 miles per hour, then it travels a total of 4 • 50, or 200 miles; if CDs cost eight dollars each, then three CDs will cost 3 • $8, or $24.

Major   Supporting   Additional   Depth Opportunities(DO)

- A third concept of multiplication is that of **multiplicative comparison**. Ex: Sara has four CDs, Joanne has three times as many as Sara, and Sylvia has half as many as Sara. Thus, Joanne has $3 \cdot 4$, or 12 CDs, and Sylvia has $\frac{1}{2} \cdot 4$, or 2 CDs.

Example for (3.OA.1):
Jim purchased 5 packages of muffins. Each package contained 3 muffins. How many muffins did Jim purchase? 5 groups of 3, 5 x 3 = 15.
Describe another situation where there would be 5 groups of 3 or 5 x 3.

Students recognize multiplication as a means to determine the total number of objects when there are a specific number of groups with the same number of objects in each group. Multiplication requires students to think in terms of groups of things rather than individual things. Students learn that the multiplication symbol 'x' means "groups of" and problems such as 5 x 7 refer to 5 groups of 7.

To further develop this understanding, students interpret a problem situation requiring multiplication using pictures, objects, words, numbers, and equations. Then, given a multiplication expression (e.g., 5 x 6) students interpret the expression using a multiplication context. (See Appendix for chart) They should begin to use the terms, *factor* and *product,* as they describe multiplication. (MP6)

Instructional Strategies: (3.OA.1-4)
In Grade 2, students found the total number of objects using rectangular arrays, such as a 5 x 5, and wrote equations to represent the sum. This strategy is a foundation for multiplication because students should make a connection between repeated addition and multiplication.

Students need to experience problem-solving involving equal groups (whole unknown or size of group is unknown) and multiplicative comparison (unknown product, group size unknown or number of groups unknown) as shown in Table 2 of Appendix.

Student should be encouraged to solve these problems in different ways to show the same idea and be able to explain their thinking verbally and in written expression. Allowing students to present several different strategies provides the opportunity for them to compare strategies.

Sets of counters, number lines to skip count and relate to multiplication and arrays/area models will aid students in solving problems involving multiplication and division. Allow students to model problems using these tools.

Students should represent the model used as a drawing or equation to find the solution.

Show a variety of models of multiplication. (i.e. 3 groups of 5 counters can be written as 3 × 5.)

Provide a variety of contexts and tasks so that students will have ample opportunity to develop and use thinking strategies to support and reinforce learning of basic multiplication and division facts.

Major   Supporting   Additional   Depth Opportunities(DO)

Ask students create multiplication problem situations in which they interpret the product of whole numbers as the total number of objects in a group. Ask them to write as an expression. Also, have students create division-problem situations in which they interpret the quotient of whole numbers as the number of shares.

Students can use known multiplication facts to determine the unknown fact in a multiplication or division problem. Have them write a multiplication or division equation and the related multiplication or division equation. For example, to determine the unknown whole number in 27 ÷ $\square = 3$, s
3 × 9 = 27. They should ask themselves questions such as, "How many 3s are in 27?" or "3 times what number is 27?" Have them justify their thinking with models or drawings.

Resources/Tools:
For detailed information see Operations and Algebraic Thinking Learning Progressions:
http://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf

See engageNY Modules

"Barnyard Legs", Georgia Department of Education. Students solve multiplication problems using different strategies based on Amanda Bean's Amazing Dream, A Mathematical Story by Cindy Neuschwander or a similar book about multiplication.

"Twenty-Four Kids All in a Row", Georgia Department of Education.

"Exploring Equal Sets", NCTM, Mathematics, Illuminations. This four-part lesson encourages students to explore models for multiplication, the inverse of multiplication, and representing multiplication facts in equation form.

"All About Multiplication", In this four-lesson unit, students explore several meanings and representation of multiplications and learn about properties of operations for multiplication.

Sets of counters

Number lines to skip count and relate to multiplication

Common multiplication and division situations Appendix , Table 2

Common Misconceptions: (3.OA.1-4)
Students think a symbol (? or []) is always the place for the answer. This is especially true when the problem is written as 15 ÷ 3 =? or 15 = $\square \times 3$.

Students also think that 3 ÷ 15 = 5 and 15 ÷ 3 = 5 are the same equations. The use of models is essential in helping students eliminate this understanding.

The use of a symbol to represent a number once cannot be used to represent another number in a different problem/situation. Presenting students with multiple situations in which they select the symbol and explain what it represents will counter this misconception.

<mark>Major</mark>　　　　<mark>Supporting</mark>　　　　<mark>Additional</mark>　　　　<mark>Depth Opportunities(DO)</mark>

Domain: Operations and Algebraic Thinking (A)

Cluster: *Represent and solve problems involving multiplication and division*.

Standard: Grade 3.OA.2

Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: See Grade 3.OA.1

Explanation and Examples:

This standard focuses on two distinct models of division: partition models and measurement (repeated subtraction) models.

- *Partition models* focus on the question, "How many in each group?" A context for partition models would be: There are 12 cookies on the counter. If you are sharing the cookies equally among three bags, how many cookies will go in each bag?
- *Measurement (repeated subtraction) models* focus on the question, "How many groups can you make?" A context or measurement models would be: There are 12 cookies on the counter. If you put 3 cookies in each bag, how many bags will you fill?

Students need to recognize the operation of division in two different types of situations. One situation requires determining how many groups and the other situation requires sharing (determining how many in each group). Students should be exposed to appropriate terminology (quotient, dividend, divisor, and factor). (MP6)

To develop this understanding, students interpret a problem situation requiring division using pictures, objects, words, numbers, and equations. Given a division expression (e.g., 24 ÷ 6) students interpret the expression in contexts that require both interpretations of division. (See Table 2 in Appendix)

Instructional Strategies: See Grade 3.OA.1

Resources/Tools

Illustrative Mathematics:

3.OA Fish Tanks

3.OA Markers in Boxes

Major   Supporting   Additional   Depth Opportunities(DO)

Common Misconceptions: See Grade 3.OA.1

Domain: Operations and Algebraic Thinking (OA)

Cluster: Represents and solve problems involving multiplication and division.

Standard: Grade 3. OA.3

Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: See Grade 3.OA.1

Explanation and Examples:

This standard references various strategies that can be used to solve word problems involving multiplication & division. Students should apply their skills to solve word problems. Students should use a variety of representations for creating and solving one-step word problems, such as: If you divide 4 packs of 9 brownies among 6 people, how many brownies does each person receive? *(4 x 9 = 36, 36 ÷ 6 = 6).*

See the Appendix, Table 2 for examples of a variety of problem solving contexts, in which students need to find the product, the group size, or the number of groups. Students should be given ample experiences to explore and make sense of ALL the different problem structures.

Examples of multiplication:

There are 24 desks in the classroom. If the teacher puts 6 desks in each row, how many rows are there? This task can be solved by drawing an array by putting 6 desks in each row. This is an array model.
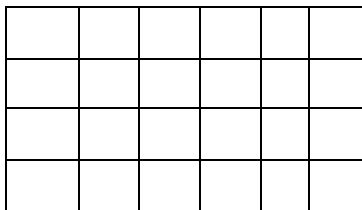
This task can also be solved by drawing pictures of equal groups. 4 groups of 6 equals 24 objects

| * * * * * * | * * * * * * | * * * * * * | * * * * * * |

A student could also reason through the problem mentally or verbally, "I know 6 and 6 are 12. 12 and 12 are 24. Therefore, there are 4 groups of 6 giving a total of 24 desks in the classroom."

Major   Supporting   Additional   Depth Opportunities(DO)

A number line could also be used to show jumps of equal *distance*.

Students in third grade students should use a variety of pictures, such as stars, boxes, flowers to represent unknown numbers (variables). Letters are also introduced to represent unknowns in third grade.

Examples of Division:
There are some students at recess. The teacher divides the class into 4 lines with 6 students in each line. Write a division equation for this story and determine how many students are in the class *(n ÷ 4 = 6. There are 24 students in the class).*

Determining the number of objects in each share (partitive division, where the size of the groups is unknown):

The bag has 92 hair clips, and Laura and her three friends want to share them equally. How many hair clips will each person receive?

Determining the number of shares (measurement division, where the number of groups is unknown)

Students use a variety of representations for creating and solving one-step word problems, i.e., numbers, words, pictures, physical objects, or equations. They use multiplication and division of whole numbers up to 10 x10. Students explain their thinking, show their work by using at least one representation, and verify that their answer is reasonable.

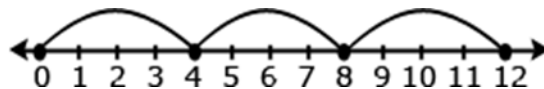Word problems may be represented in multiple ways:
- Equations: 3 x 4 = ?, 4 x 3 = ?, 12 ÷ 4 = ? and 12 ÷ 3 = ?
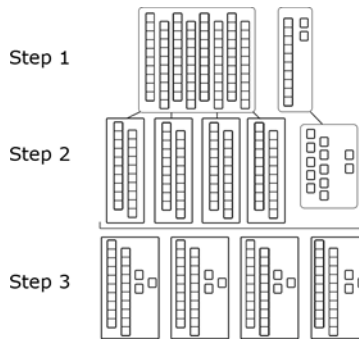- Array:

- Equal groups

- Repeated addition: 4 + 4 + 4 or repeated subtraction
- Three equal jumps (distances) forward from 0 on the number line to 12 or three equal jumps (distances) backwards from 12 to 0

Examples of division problems:

- Determining the number of objects in each share (partitive division, where the size of the groups is unknown):
    - o The bag has 92 hair clips, and Laura and her three friends want to share them equally. How many hair clips will each person receive?

- Determining the number of shares (measurement division, where the number of groups is unknown)
    - o Max the monkey loves bananas. Molly, his trainer, has 24 bananas. If she gives Max 4 bananas each day, how many days will the bananas last?

Starting	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
24	24 − 4 = 20	20 − 4 = 16	16 − 4 = 12	12 − 4 = 8	8 − 4 = 4	4 − 4 = 0

Solution: The bananas will last for 6 days.

Instructional Strategies: See Grade 3.OA.1

Tools/Resources:

Illustrative Math Tasks:

3.OA Two Interpretations of Division

3.OA Analyzing Word Problems Involving Multiplication

3.OA Gifts from Grandma, Variation 1

3.OA, MD, NBT Classroom Supplies

For detailed information see: Learning Progressions- Operations and Algebraic Thinking K-5

Common Misconceptions: See Grade 3.OA.1

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Operations and Algebraic Thinking (OA)

Cluster: Represent and solve problems involving multiplication and division.

Standard: Grade 3.OA. 4

Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = □ ÷ 3, 6 × 6 = ?*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: See Grade 2.OA.1

Explanation and Examples:

This standard refers to Table 2 in the Appendix of this document and equations for the different types of multiplication and division problem structures. The easiest problem structure includes Unknown Product (3 x 6 = ? or 18 ÷ 3 = 6). The more difficult problem structures include Group Size Unknown (3 x ? = 18 or 18 ÷ 3 = 6) or Number of Groups Unknown (? x 6 = 18, 18 ÷ 6 = 3).

The focus of 3.OA.4 goes beyond the traditional notion of *fact families*, by having students explore the **inverse relationship** of multiplication and division.

Students apply their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given 4 x ? = 40, they might think:

- 4 groups of some number is the same as 40

Students apply their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given 4 x ? = 40, they might think:

- 4 groups of some number is the same as 40
- 4 times some number is the same as 40
- I know that 4 groups of 10 is 40 so the unknown number is 10
- The missing factor is 10 because 4 times 10 equals 40.

Equations in the form of a x b = c and c = a x b should be used interchangeably, with the unknown in different positions.

Example:

Solve the equations below:

$$24 = ? \, X \, 6$$
$$72 \div \blacksquare = 9$$

Melisa has 3 bags. There are 4 marbles in each bag. How many marbles does Melisa have altogether? $3 \times 4 = m$

This standard is strongly connected to 3.OA.3 when students solve problems and determine unknowns in equations.

Students should also experience creating story problems for given equations. When crafting story problems, they should carefully consider the question(s) to be asked and answered to write an appropriate equation.

Students may approach the same story problem differently and write either a multiplication equation or division equation*.*

Students apply their understanding of the meaning of the equal sign as" the same value as" to interpret an equation with an unknown. When given $4 \times ? = 40$, they might think:

- 4 groups of some number is the same as 40
- 4 times some number is the same as 40
- I know that 4 groups of 10 is 40 so the unknown number is 10
- The missing factor is 10 because 4 times 10 equals 40.

Equations in the form of $a \times b = c$ and $c = a \times b$ should be used interchangeably, with the unknown in different positions.

Instructional Strategies: **See Grade 3.OA.1**

Resources/Tools:
Illustrative Mathematics:
3.OA Finding the unknown in a division equation

Common Misconceptions: See Grade 3.OA.1

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Operations and Algebraic Thinking (OA)

Cluster: Understand properties of multiplication and the relationship between multiplication and division.

Standard: Grade 3.OA.5

Apply properties of operations as strategies to multiply and divide. (Students need not use formal terms for these properties.) *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.)* $3 \times 5 \times 2$ *can be found by* $3 \times 5 = 15$*, then* $15 \times 2 = 30$*, or by* $5 \times 2 = 10$*, then* $3 \times 10 = 30$*. (Associative property of multiplication.) Knowing that* $8 \times 5 = 40$ *and* $8 \times 2 = 16$*, one can find* 8×7 *as* $8 \times (5 + 2 = (8 \times 5) + (8 \times 2) = 40 + 16 = 56)$*. (Distributive property.)*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.6 Attend to precision
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (Grade 3.OA.5-6)

This cluster is connected to:

- Third Grade Critical Area of Focus #1, **Developing understanding of multiplication and division and strategies for multiplication and division within 100**.

Explanation and Examples:

This standard references properties of multiplication. While students DO NOT need to use the formal terms of these properties, student should understand that properties are rules about how numbers work.

Students do need to be flexible and fluent applying each of them. Students represent expressions using various objects, pictures, words and symbols in order to develop their understanding of properties. They multiply by 1 and 0 and divide by 1. They change the order of numbers to determine that the order of numbers does not make a difference in multiplication (**but does make a difference in division**).

Given three factors, they investigate how changing the order of how they multiply the numbers does not change the product. They also decompose numbers to build fluency with multiplication.

The *associative property* states that the sum or product stays the same when the grouping of addends or factors is changed. For example, when a student multiplies 7 x 5 x 2, a student could rearrange the numbers to first multiply 5 x 2 = 10 and then multiply 10 x 7 = 70.

The *commutative property* (order property) states that the order of numbers does not matter when adding or multiplying numbers. For example, if a student knows that 5 x 4 = 20, then they also know that 4 x 5 = 20.

Major   Supporting   Additional   Depth Opportunities(DO)

The array below could be described as a 5 x 4 array for 5 columns and 4 rows, or a 4 x 5 array for 4 rows and 5 columns. **There is no "fixed" way to write the dimensions of an array as rows x columns or columns x rows.**

Students should have flexibility in being able to describe both dimensions of an array.

```
X X X X          X X X X X
X X X X    4x5   X X X X X
X X X X     or   X X X X X
X X X X    5x4   X X X X X
X X X X
```

Students should be introduced to the *distributive property* of multiplication over addition as a strategy for using products they know to solve products they don't know. Students would be using mental math to determine a product.

Here are ways that students could use the distributive property to determine the product of 7 x 6. Again, students should use the distributive property, but can refer to this in informal language such as "breaking numbers apart".

Student 1	Student 2	Student 3
7 x 6	7 x 6	7 x 6
7 x 5 = 35	7 x 3 = 21	5 x 6 = 30
7 x 1 = 7	7 x 3 = 21	2 x 6 = 12
35 + 7 = 42	21 + 21 = 42	30 + 12 = 42

Another example of the distributive property helps students determine the products and factors of problems by breaking numbers apart. For example, for the problem 6 x 5= ?, students can decompose the 6 into a 4 and 2, and reach the answer by multiplying 4 x 5 = 20 and 2 x 5 =10 and adding the two products (20+10=30).
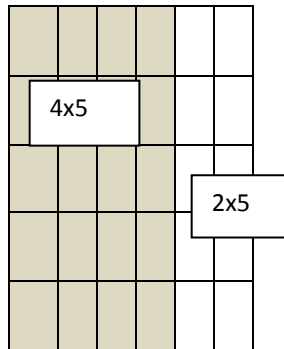
To further develop understanding of properties related to multiplication and division, students use different representations and their understanding of the relationship between multiplication and division to determine if the following types of equations are true or false.

$$0 \times 7 = 7 \times 0 = 0 \text{ (Zero Property of Multiplication)}$$
$$1 \times 9 = 9 \times 1 = 9 \text{ (Multiplicative Identity Property of 1)}$$
$$3 \times 6 = 6 \times 3 \quad \text{(Commutative Property)}$$
$$8 \div 2 \neq 2 \div 8 \quad \text{(Students are only to determine that these are not equal)}$$
$$2 \times 3 \times 5 = 6 \times 5$$
$$10 \times 2 < 5 \times 2 \times 2$$
$$2 \times 3 \times 5 = 10 \times 3$$
$$1 \times 6 > 3 \times 0 \times 2$$

Students represent expressions using various objects, pictures, words and symbols in order to develop their understanding of properties. They multiply by 1 and 0 and divide by 1, never by 0. They change the order of numbers to determine that the order of numbers does not make a difference in multiplication (but does make a difference in division).

Given three factors, they investigate changing the order of how they multiply the numbers to determine that changing the order does not change the product. They also decompose numbers to build fluency with multiplication.

Use models to help build understanding of the commutative property:

Example: 3 x 6 = 6 x 3

In the following diagram it may not be obvious that 3 groups of 6 is the same as 6 groups of 3. A student may need to count to verify this.

 is the same quantity as
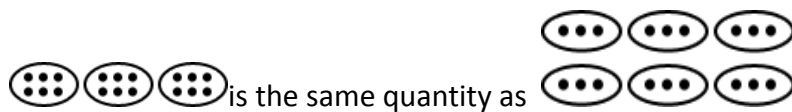
Different representation:
An array explicitly demonstrates the concept of the commutative property.

4 rows of 3 or 4 x 3   3 rows of 4 or 3 x 4

Students are introduced to the distributive property of multiplication over addition as a strategy for using products they know to solve products they don't know. For example, if students are asked to find the product of 7 x 8, they might decompose 7 into 5 and 2 and then multiply 5 x 8 and 2 x 8 to arrive at 40 + 16 or 56.

Students should learn that they can decompose either of the factors. It is important to note that the students may record their thinking in different ways.

$5 \times 8 = 40$
$2 \times 8 = \underline{16}$
$ 56$

$5 \times 8 = 40$

$2 \times 8 = 16$

$7 \times 4 = 28$
$7 \times 4 = \underline{28}$
$ 56$
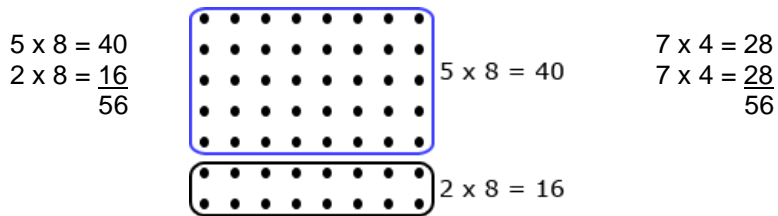
Instructional Strategies: (3.OA.5-6)

Students need to apply properties of operations (commutative, associative and distributive) as strategies to multiply and divide. Applying the concept involved is more important than students knowing the name of the property.

Understanding the commutative property of multiplication is developed through the use of models as basic multiplication facts are learned. For example, the result of multiplying 3 x 5 (15) is the same as the result of multiplying 5 x 3 (15).

Splitting arrays can help students understand the distributive property. They can use a known fact to learn other facts that may cause difficulty. (See example above where students split an array into smaller arrays and add the sums of the groups.)

Students' understanding of the part/whole relationships is critical in understanding the connection between multiplication and division.

Resources/Tools

"Multiplication--It's In the Cards", Illuminations, NCTM. Students skip-count and examine multiplication patterns. They also explore the commutative property of multiplication.

"Multiplication--It's In the Cards: Looking for Calculator Patterns", Illuminations, NCTM. Students use a web-based calculator to create and compare counting patterns using the constant function feature of the calculator. Making connections between multiple representations of counting patterns reinforces students understanding of this important idea and helps them recall these patterns as multiplication facts. From a chart, students notice that multiplication is commutative.

Illustrative Mathematics:
3.OA Valid Equalities? (Part 2)

Major   Supporting   Additional   Depth Opportunities(DO)

Common Misconceptions:

Students may experience difficulty in determining which factor represents rows or the number of objects in a group, and which factor represents the number of groups or columns. In division there are two different situations that can cause confusion depending on which factor is the unknown—the number in the group or the number of groups.

Domain: Operations and Algebraic Thinking (OA)

Standard: Grade 3.OA.6

Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1   Make sense of problems and persevere in solving them.
- ✓ MP.7   Look for and make use of structure.

Connections: See Grade 3.OA.5

Explanation and Examples:

This standard refers the Table 2 in the Appendix. Since multiplication and division are inverse operations, students are expected to solve problems and explain their processes of solving division problems that can also be represented as unknown factor in multiplication problems.

Example:

A student knows that 2 x 9 = 18. How can they use that fact to determine the answer to the following question: 18 people are divided into pairs in P.E. class? How many pairs are there? Write a division equation and explain your reasoning.

Multiplication and division are inverse operations and that understanding can be used to find the unknown. Fact family triangles demonstrate the inverse operations of multiplication and division by showing the two factors and how those factors relate to the product and/or quotient.

Example:

- 3 x 5 = 15   5 x 3 = 15
- 15 ÷ 3 = 5   15 ÷ 5 = 3
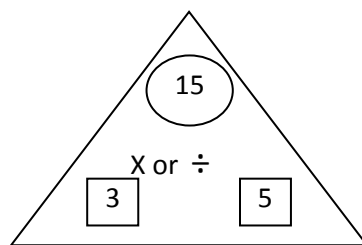
Multiplication and division are inverse operations and that understanding can be used to find the unknown. Fact family triangles demonstrate the inverse operations of multiplication and division by showing the two factors and how those factors relate to the product and/or quotient.

Major   Supporting   Additional   Depth Opportunities(DO)

Students use their understanding of the meaning of the equal sign as "the same value as" to interpret an equation with an unknown. When given $32 \div \blacksquare = 4$, students may think:

- 4 groups of some number is the same as 32
- 4 times some number is the same as 32
- I know that 4 groups of 8 is 32 so the unknown number is 8
- The missing factor is 8 because 4 times 8 is 32.

Equations in the form of $a \div b = c$ and $c = a \div b$ need to be used interchangeably, with the unknown in different positions.

Instructional Strategies: **See Grade 3. OA.5**

Common Misconceptions: See Grade 3. OA.5

Domain: Operations and Algebraic Thinking (OA)

Cluster: <mark>**_Multiply and divide within 100._**</mark>

Standard: Grade 3. OA.7

<mark>Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.</mark>

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Third Grade <mark>Critical Area of Focus #1, **Developing understanding of multiplication and division and strategies for multiplication and division within 100**</mark>.

Explanation and Examples:

This standard uses the word fluently, which means accuracy, efficiency (**using a reasonable amount of steps and time**), and flexibility (using strategies such as the distributive property). **"Know from memory" does not mean focusing only on timed tests and repetitive practice,** but ample experiences working with manipulatives, pictures, arrays, word problems, and numbers to internalize the basic facts (up to 9 x 9).

By studying patterns and relationships in multiplication facts and relating multiplication and division, students build a foundation for fluency with multiplication and division facts. Students demonstrate fluency with multiplication facts through 10 and the related division facts. Multiplying and dividing fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

Strategies students may use to attain fluency include:

- Multiplication by zeroes and ones
- Doubles (2s facts), Doubling twice (4s), Doubling three times (8s)
- Tens facts (relating to place value, 5 x 10 is 5 tens or 50)
- Five facts (half of tens)
- Skip counting (counting groups of __ and knowing how many groups have been counted)
- Square numbers (ex: 3 x 3)
- Nines (10 groups less one group, e.g., 9 x 3 is 10 groups of 3 minus one group of 3)
- Decomposing into known facts (6 x 7 is 6 x 6 plus one more group of 6)
- Turn-around facts (Commutative Property)
- Fact families (Ex: 6 x 4 = 24; 24 ÷ 6 = 4; 24 ÷ 4 = 6; 4 x 6 = 24)
- Missing factors

<mark>Major</mark>   <mark>Supporting</mark>   <mark>Additional</mark>   <mark>Depth Opportunities(DO)</mark>

General Note: Students should have exposure to multiplication and division problems presented in both vertical and horizontal forms. *(Problems presented horizontally encourage solving mentally.)*

By studying patterns and relationships in multiplication facts and relating multiplication and division, students build a foundation for fluency with multiplication and division facts. Students demonstrate fluency with multiplication facts through 10 and the related division facts. Multiplying and dividing fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

Instructional Strategies:

Students need to understand the part/whole relationships in order to understand the connection between multiplication and division. They need to develop efficient strategies that lead to the big ideas of multiplication and division.

- These big ideas include understanding the properties of operations, such as the commutative and associative properties of multiplication and the distributive property. The naming of the property is not necessary at this stage of learning.
- In Grade 2, students found the total number of objects using rectangular arrays, such as a 5 x 5, and wrote equations to represent the sum. **This is called unitizing**. It requires students to count groups, not just objects. They see the whole as a number of groups of a number of objects. This strategy is a foundation for multiplication in that students should make a connection between repeated addition and multiplication.

As students create arrays for multiplication using objects or drawing on graph paper, they may discover that three groups of four and four groups of three yield the same results.

They should observe that the arrays stay the same, although how they are viewed changes. Provide numerous situations for students to develop this understanding.
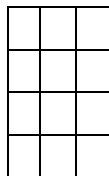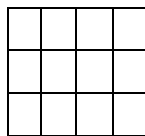
To develop an understanding of the distributive property, students need decompose the whole into groups. Arrays can be used to develop this understanding. To find the product of 3 × 9, students can decompose 9 into the sum of 4 and 5 and find 3 × (4 + 5).
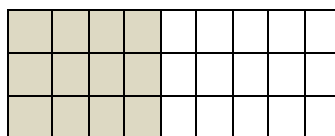
Major   Supporting   Additional   Depth Opportunities(DO)

The distributive property is the basis for the standard multiplication algorithm that students can use to fluently multiply multi-digit whole numbers in Grade 5.

Once students have an understanding of multiplication using efficient strategies, they should make the connection to division.

Using various strategies to solve different contextual problems that use the same two one-digit whole numbers requiring multiplication allows for students to commit to memory all products of two one-digit numbers.

Resources/Tools:
Unifix cubes
Grid or graph paper
Sets of counters

See: K-5 Operations and Algebraic Thinking and Counting and Cardinality for detailed information.

"A Giraffe Named Stretch", Georgia Department of Education. Students create and solve multiplication stories about Stretch (a giraffe) and his children using a list of facts given to them.

"Making Sense of Division", Georgia Department of Education. Students demonstrate how to use division as an application of money. Students observe what happens when an amount of money is divided evenly among a group of people or not divided evenly among a group of people.

Common Misconceptions:
Student who struggle most likely do not have fluency for the easy numbers. The child does not understand an unknown factor (a divisor) can be found from the related multiplication. It is not a matter of instilling facts divorced from their meaning, but rather the outcome of carefully designed learning. That involves the interplay of practice and reasoning.

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Operations and Algebraic Thinking (OA)

Cluster: Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Standard: Grade 3.OA.8

Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. (This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

- Third Grade Critical Area of Focus #1, **Developing understanding of multiplication and division and strategies for multiplication and division within 100**.
- Represent and solve problems involving multiplication and division. (Grade 3 OA 1 – 4)
- Use place value understanding and properties of operations to perform multi-digit arithmetic. (Grade 3 NBT 1-3)

Explanation and Examples:

This standard refers to two-step word problems using the four operations. The size of the numbers should be limited to related 3rd grade standards (e.g., 3.OA.7 and 3.NBT.2). Adding and subtracting numbers should include numbers within 1,000, and multiplying and dividing numbers should include single-digit factors and products less than 100.

This standard calls for students to represent problems using equations with a letter to represent unknown quantities.

Example:
Mike runs 2 miles a day. His goal is to run 25 miles. After 5 days, how many miles does Mike have left to run in order to meet his goal? Write an equation and find the solution (2 x 5 + m = 25).

This standard refers to estimation strategies, including using compatible numbers (numbers that sum to 10, 50, or 100) or rounding. The focus in this standard is to have students use and discuss various strategies. Students should estimate during problem solving, and then revisit their estimate to check for reasonableness.

Major   Supporting   Additional   Depth Opportunities(DO)

Example:

Here are some typical estimation strategies for the problem:

On a vacation, your family travels 267 miles on the first day, 194 miles on the second day and 34 miles on the third day. How many total miles did they travel?

Student 1	Student 2	Student 3
I first thought about 267 and 34. I noticed that their sum is about 300. Then I knew that 194 is close to 200. When I put 300 and 200 together, I get 500.	I first thought about 194. It is really close to 200. I also have 2 hundreds in 267. That gives me a total of 4 hundreds. Then I have 67 in 267 and the 34. When I put 67 and 34 together that is really close to 100. When I add that hundred to the 4 hundreds that I already had, I end up with 300.	I rounded 267 to 300. I rounded 194 to 200. I rounded 34 to 30. When I added 300, 200 and 30, I know my answer will be about 530.

The assessment of estimation strategies should only have one reasonable answer (500 or 530), or a range (between 500 and 550). Problems should be structured so that all acceptable estimation strategies will arrive at a reasonable answer. Students should be expected to explain their thinking in arriving at the answer.

It is important that students be exposed to multiple problem-solving strategies (using any combination of words, numbers, diagrams, physical objects or symbols) and be able to choose which ones to use.

Examples:

- Jerry earned 231 points at school last week. This week he earned 79 points. If he uses 60 points to earn free time on a computer, how many points will he have left?
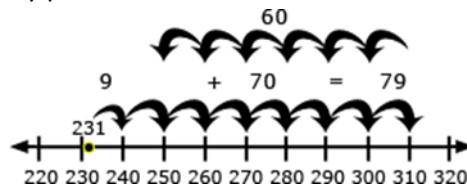
A student may use the number line above to describe his/her thinking, "231 + 9 = 240 so now I need to add 70 more. 240, 250 (10 more), 260 (20 more), 270, 280, 290, 300, 310 (70 more). Now I need to count back 60. 310, 300 (back 10), 290 (back 20), 280, 270, 260, 250 (back 60)."

A student writes the equation, 231 + 79 – 60 = m and uses rounding (230 + 80 – 60) to estimate.

A student writes the equation, 231 + 79 – 60 = m and calculates 79-60 = 19 and then calculates 231 + 19 = m.

- The soccer club is going on a trip to the water park. The cost of attending the trip is $63. Included in that price is $13 for lunch and the cost of 2 wristbands, one for the morning and one for the afternoon. Write an equation representing the cost of the field trip and determine the price of one wristband.

w	w	13
63		

The above diagram helps the student write the equation, w + w + 13 = 63. Using the diagram, a student might think, "I know that the two wristbands cost $50 ($63-$13) so one wristband costs $25." To check for reasonableness, a student might use front end estimation and say 60-10 = 50 and 50 ÷ 2 = 25.

Student should use various estimation skills solve word problems. They should include:
- identifying when estimation is appropriate
- determining the level of accuracy needed
- selecting the appropriate method of estimation
- verifying solutions or determining the reasonableness of solutions.

Estimation strategies include, but are not limited to:
- using benchmark numbers that are easy to compute
- front-end estimation with adjusting:
  1. (using the highest place value and estimating from the front end making adjustments to the estimate by taking into account the remaining amounts)
  2. rounding and adjusting (students round down or round up and then adjust their estimate depending on how much the rounding changed the original values)

Instructional Strategies: **(3.OA.8-9)**
Students gain a full understanding of which operation to use in any given situation through contextual problems. Number skills and concepts are developed as students solve problems. Problems should be presented on a regular basis as students work with numbers and computations.

Researchers and mathematics educators advise against providing "key words" for students to look for in problem situations because they can be misleading. Students should use various strategies to solve problems. Students should analyze the structure of the problem to make sense of it. They should think through the problem and the meaning of the answer before attempting to solve it. (M. Burns)

Encourage students to represent the problem situation in a drawing or with counters or blocks. Students should determine the reasonableness of the solution to all problems using mental computations and estimation strategies.

Students can use base–ten blocks on centimeter grid paper to construct rectangular arrays to represent problems.

Students are to identify arithmetic patterns and explain the patterns using properties of operations. They can explore patterns by determining likenesses, differences and changes. Use patterns in addition and multiplication tables.

Major   Supporting   Additional   Depth Opportunities(DO)

Resources/Tools:
"Multiplication--It's In the Cards", Illuminations, NCTM. Students skip-count and examine multiplication patterns. They also explore the commutative property of multiplication.

"Multiplication--It's In the Cards: Looking for Calculator Patterns", Illuminations, NCTM.   Students use a web-based calculator to create and compare counting patterns using the constant function feature of the calculator. Making connections between multiple representations of counting patterns reinforces students understanding of this important idea and helps them recall these patterns as multiplication facts. From a chart, students notice that multiplication is commutative.

Illustrative Mathematics:
3.OA The Stamp Collection
3.OA The Class Trip

Common Misconceptions: See 3.OA.7

Domain: Operations and Algebraic Thinking (OA)

Standard: Grade 3.OA. 9

Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See Grade 3. OA.8

Explanation and Examples:

This standard calls for students to examine arithmetic patterns involving both addition and multiplication.

Arithmetic patterns are patterns that change by the same rate, such as adding the same number. For example, the series 2, 4, 6, 8, 10 is an arithmetic pattern that increases by 2 between each term.

This standard also mentions identifying patterns related to the properties of operations.

Examples:

- Even numbers are always divisible by 2. Even numbers can always be decomposed into 2 equal addends (14 = 7 + 7).
- Multiples of even numbers (2, 4, 6, and 8) are always even numbers.
- On a multiplication chart, the products in each row and column increase by the same amount (skip counting).
- On an addition chart, the sums in each row and column increase by the same amount.
- Using a multiplication table, highlight a row of numbers and ask students what they notice about the highlighted numbers.

Explain a pattern using properties of operations.

When (commutative property) one changes the order of the factors they will still get the same product, example 6 x 5 = 30 and 5 x 6 = 30.

Teacher: What pattern do you notice when 2, 4, 6, 8, or 10 are multiplied by any number (even or odd)?

Student: The product will always be an even number.

Major   Supporting   Additional   Depth Opportunities(DO)

Teacher: Why?

In an addition table ask what patterns they notice. Explain why the pattern works this way?

Students need ample opportunities to observe and identify important numerical patterns related to operations. They should build on their previous experiences with properties related to addition and subtraction. Students investigate addition and multiplication tables in search of patterns and explain why these patterns make sense mathematically. (MPs 7&8).

*All of the **understandings** of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single digit numbers and 10.*

It should be clear, this does not mean instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involved the interplay of PRACTICE and REASONING. (Learning Progressions-Operations and Algebraic Thinking K-5).

Examples:
- Any sum of two even numbers is even.
- Any sum of two odd numbers is even.
- Any sum of an even number and an odd number is odd.
- The multiples of 4, 6, 8, and 10 are all even because they can all be decomposed into two equal groups.
- The doubles (2 addends the same) in an addition table fall on a diagonal while the doubles (multiples of 2) in a multiplication table fall on horizontal and vertical lines.
- The multiples of any number fall on a horizontal and a vertical line due to the commutative property.
- All the multiples of 5 end in a 0 or 5 while all the multiples of 10 end with 0. Every other multiple of 5 is a multiple of 10.

Students also investigate a hundreds chart in search of addition and subtraction patterns. They record and organize all the different possible sums of a number and explain why the pattern makes sense.

Addend	Addend	Sum
0	20	20
1	19	20
2	18	20
3	17	20
4	16	20
?	?	?
?	?	?
?	?	?
20	0	20

Major   Supporting   Additional   Depth Opportunities(DO)

Instructional Strategies: See Grade 3.OA.8

Resources/Tools:
Illustrative Mathematics:
[3.OA Addition Patterns](#)
[3.OA Patterns in the multiplication table](#)
[3.OA Symmetry of the addition table](#)
[3.OA Making a ten](#)

Common Misconceptions:
The student is not able to follow the conventions of order of operations. They randomly attack pairs of numbers without regard for what the associative and distributive properties require. They do not look for and make use of structure (MP7) or they do not follow the "rules of the road".

Domain: Number and Operations in Base Ten (NBT)

Cluster: *Use place value understanding and properties of operations to perform multi-digit arithmetic.*

Standard: Grade 3.NBT.1

Use place value understanding to round whole numbers to the nearest 10 or 100.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: (3.NBT.1-3)

This cluster is connected to:

- Third Grade Critical Area of Focus #1, **Developing understanding of multiplication and division and strategies for multiplication and division within 100**.
- Additionally, the content in this cluster goes beyond the critical areas to address solving multi-step problems.
- The rounding strategies developed in third grade will be expanded in grade four with larger numbers.
- Additionally, students will formalize the rules for rounding numbers with the expansion of numbers in fourth grade.
- In fourth grade the place value concepts developed in grades K-3 will be expanded to include decimal notation.
- Understand place value. (Grade 2 NBT 1 – 4 and Grade 2 NBT 5 – 9)

Explanation and Examples:

This standard refers to place value understanding, which extends beyond an algorithm or procedure for rounding.

The expectation is that students have a deep understanding of place value and number sense and can explain and reason about the answers they get when they round.

Students should have numerous experiences using a number line and a hundreds chart as tools to support their work with rounding. Students learn when and **why** to round numbers. They identify possible answers and halfway points. Then they narrow where the given number falls between the possible answers and halfway points. They also understand that by convention if a number is exactly at the halfway point of the two possible answers, at this level the number is rounded up.
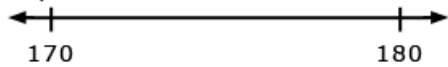
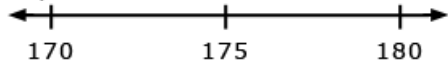Major   Supporting   Additional   Depth Opportunities(DO)

Example: Round 178 to the nearest 10.
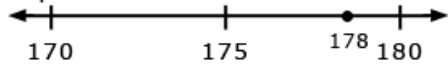
Step 1

Step 1: The answer is either 170 or 180.
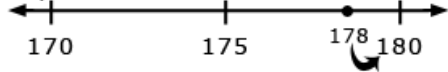
Step 2

Step 2: The halfway point is 175.

Step 3

Step 3: 178 is between 175 and 180.

Step 4

Step 4: Therefore, the rounded number is 180.

Instructional Strategies:

Prior to implementing rules for rounding students need to have opportunities to investigate place value. A strong understanding of place value is essential for the development of number sense and the subsequent work that involves rounding numbers.

Building on previous understandings of the place value of digits in multi-digit numbers, place value is used to round whole numbers. Dependence on learning rules can be eliminated with strategies such as the use of a number line to determine which multiple of 10 or of100, a number is nearest (5 or more rounds up, less than 5 rounds down). As students' understanding of place value increases, the strategies for rounding are valuable for estimating, justifying and predicting the reasonableness of solutions in problem-solving.

Strategies used to add and subtract two-digit numbers can now be applied to fluently to add and subtract whole numbers within 1000. These strategies should be discussed so that students can make comparisons and move toward efficient methods.

Number sense and computational understanding is built on a firm understanding of place value.

Understanding what each number in a multiplication expression represents is important. Multiplication problems need to be modeled with pictures, diagrams or concrete materials to help students understand what the factors and products represent. The effect of multiplying numbers needs to be examined and understood.

The use of area models is important in understanding the properties of operations of multiplication and the relationship of the factors and its product. Composing and decomposing area models is useful in the development and understanding of the distributive property in multiplication.
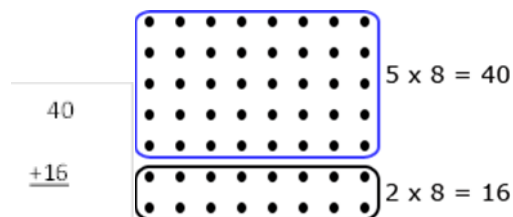
Continue to use manipulative like hundreds charts and place-value charts. Have students use a number line or a roller coaster example to block off the numbers in different colors.

For example this chart show what numbers will round to the tens place.

Round	Round	Round	Round	Round
to 0	to 10	to 20	to 30	to 40
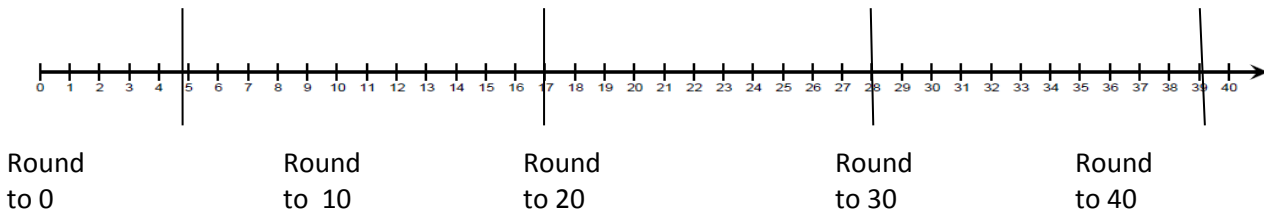
Tools / Resources:
See Learning Progressions NBT for detailed information:

Illustrative Mathematics Tasks:
Illustrative Mathematics:

3.NBT Rounding to 50 or 500
3.NBT Rounding to the Nearest Ten and Hundred
3.NBT, 4.NBT Rounding to the Nearest 100 and 1000

Also see: "Correcting the Calculator," NCSM, Great Tasks for Mathematics K-5, (2013).

Common Misconceptions: (3.NBT.1-3)
The use of terms like "round up" and "round down" confuses many students. For example, the number 37 would round to 40 or they say it "rounds up". The digit in the tens place is changed from 3 to 4 (rounds up). This misconception is what causes the problem when applied to rounding down.

The number 32 should be rounded (down) to 30, but using the logic mentioned for rounding up, some students may look at the digit in the tens place and take it to the previous number, resulting in the incorrect value of 20.

To remedy this misconception, students need to use a number line to visualize the placement of the number and/or ask questions such as: "What tens are 32 between and which one is it closer to?"

Developing the **understanding** of the **WHY** behind rounding, what the answer choices are, using place value understanding to round numbers, rather than relying on rounding rhymes e.g. *Find your number, look next door, fiver or greater add on one more, and so on,* can alleviate much of the misconception and confusion related to rounding.

Domain: Number and Operations in Base Ten (NBT)

(A range of algorithms may be used)

Standard: Grade 3.NBT.2
Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.2   Reason abstractly and quantitatively.
- ✓ MP.7   Look for and make use of structure.
- ✓ MP.8   Look for and express regularity in repeated reasoning.

Connections:
See Grade 3.NBT.1

Explanation and Examples:
This standard refers to fluently, which means accuracy, efficiency (using a reasonable number of steps and time), and flexibility (using strategies such as the distributive property). The word algorithm refers to a procedure or a series of steps. There are other algorithms other than the standard/traditional algorithm. Third grade students should have experiences beyond the standard/traditional algorithm.

Problems should include both vertical and horizontal forms, including opportunities for students to apply the commutative and associative properties. Students explain their thinking and show their work by using strategies and algorithms, and verify that their answer is reasonable.

Example:
There are 178 fourth graders and 225 fifth graders on the playground. What is the total number of students on the playground?

Student 1	Student 2	Student 3	Student 4
$100 + 200 = 300$ $70 + 20 = 90$ $8 + 5 = 13$ $300 + 90 + 13 = 403 \; students$	I added 2 to 178 to get 180. I added 220 to get 400. I added the 3 left over to get 403.	I know the 75 plus 25 equals 100. I then added 1 hundred from 178 and 2 hundreds from 275. I had a total of 4 hundreds and I had 3 more left to add. So I have 4 hundreds plus 3 more which is 403.	$178 + 225 =?$ $178 + 200 = 378$ $378 + 20 = 398$ $398 + 5 = 403$

Problems should include both vertical and horizontal forms, including opportunities for students to apply the commutative and associative properties. **Adding and subtracting fluently refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.** Students explain their thinking and show their work by using strategies and algorithms, and verify that their answer is reasonable. An interactive whiteboard or document camera may be used to show and share student thinking.

Major   Supporting   Additional   Depth Opportunities(DO)

Example:

- Mary read 573 pages during her summer reading challenge. She was only required to read 399 pages. How many extra pages did Mary read beyond the challenge requirements?

Students may use several approaches to solve the problem including the traditional algorithm. Examples of other methods students may use are listed below:

- 399 + 1 = 400, 400 + 100 = 500, 500 + 73 = 573, therefore 1+ 100 + 73 = 174 pages (Adding up strategy)
- 400 + 100 is 500; 500 + 73 is 573; 100 + 73 is 173 plus 1 (for 399, to 400) is 174 (Compensating strategy)
- Take away 73 from 573 to get to 500, take away 100 to get to 400, and take away 1 to get to 399. Then 73 +100 + 1 = 174 (Subtracting to count down strategy)
- 399 + 1 is 400, 500 (that's 100 more). 510, 520, 530, 540, 550, 560, 570, (that's 70 more), 571, 572, 573 (that's 3 more) so the total is 1 + 100 + 70 + 3 = 174 (Adding by tens or hundreds strategy)

Instructional Strategies:

See Grade 3.NBT.1

Resources/Tools:

3.OA, MD, NBT Classroom Supplies

Common Misconceptions:

See Grade 3.NBT.1

Domain: Number and Operations in Base Ten (NBT)

Standard: Grade 3.NBT.3
Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:
See Grade 3.NBT.1

Explanation and Examples:
This standard extends students' work in multiplication by having them apply their understanding of place value.
This standard expects that students go beyond tricks that hinder understanding such as "just adding zeroes" and explain and reason about their products. For example, for the problem 50 x 4, students should think of this as 4 groups of 5 tens or 20 tens. Twenty tens equals 200.

Students use base ten blocks, diagrams, or hundreds charts to multiply one-digit numbers by multiples of 10 from 10-90. They apply their understanding of multiplication and the meaning of the multiples of 10. For example, 30 is 3 tens and 70 is 7 tens. They can interpret 2 x 40 as 2 groups of 4 tens or 8 groups of ten. They understand that 5 x 60 is 5 groups of 6 tens or 30 tens and know that 30 tens is 300. After developing this understanding they begin to recognize the patterns in multiplying by multiples of 10.

Students may use manipulatives, drawings, document camera, or interactive whiteboard to demonstrate their understanding.

Instructional Strategies:
See Grade 3.NBT.1

Resources/Tools:
See EngageNY Modules

Illustrative Mathematics Tasks:
3.NBT How Many Colored Pencils?

Common Misconceptions:
See Grade 3.NBT.1

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Number and Operations—Fractions (NF)

Standard: Grade 3.NF.1

Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Third Grade Critical Area of Focus #2, Developing understanding of fractions, especially unit fractions (fractions with numerator 1).
- Partitioning traditional shapes into equal parts. (Grade 1 G 3)

## Explanation and Examples:

Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

This standard refers to the sharing of a whole being partitioned or split. Fraction models in third grade include area (parts of a whole) models (circles, rectangles, squares) and number lines.

Set models (parts of a group) are not explored in Third Grade. In 3.NF.1 students should focus on the concept that a fraction is made up (composed) of many pieces of a **unit fraction**, which has a numerator of 1. For example, the fraction $\frac{3}{5}$ is composed of 3 pieces that each have a size of $\frac{1}{5}$.

Some important concepts related to developing understanding of fractions include:

- Understand fractional parts must be equal-sized

**Example**   **Non-example**
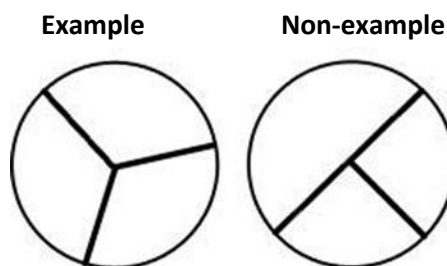
- The number of equal parts tells how many make a whole.
- As the number of equal pieces in the whole increases, the size of the fractional pieces decreases.
- The size of the fractional part is relative to the whole.

To develop understanding of fair shares, students first participate in situations where the number of objects is greater than the number of children and then progress into situations where the number of objects is less than the number of children.

Examples: (*Area or Region Model*)
- Four children share six brownies so that each child receives a fair share. How many brownies will each child receive?
- Six children share four brownies so that each child receives a fair share. What portion of each brownie will each child receive?
- What fraction of the rectangle is shaded? How might you draw the rectangle in another way but with the same fraction shaded?
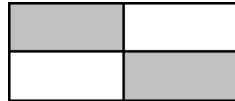
Solution: $\dfrac{2}{4}$ or $\dfrac{1}{2}$

What fraction does the letter **a** represent? (Linear Model) Explain your thinking.

a
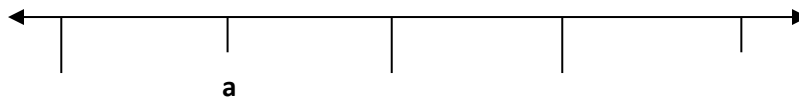
Instructional Strategies: (3.NF.1-3)
This is the initial experience students will have with fractions and instruction is best implemented over time. Students need many opportunities to discuss fractional parts using concrete models to develop familiarity and understanding of fractions. Expectations in this domain are limited to fractions with denominators 2, 3, 4, 6 and 8.

Understanding that a fraction is a quantity formed by part of a whole is essential to number sense with fractions. Fractional parts are the building blocks for all fraction concepts. Students need to relate dividing a shape into equal parts and representing this relationship on a number line, where the equal parts are between two whole numbers.

Help students plot fractions on a number line, by using the meaning of the fraction. For example, to plot $\dfrac{4}{5}$ on a number line, there are 5 equal parts with 4 copies of the 5 equal parts.

5 equal parts make the whole

$0 \qquad \dfrac{1}{5} \qquad \dfrac{2}{5} \qquad \dfrac{3}{5} \qquad \dfrac{4}{5} \qquad \dfrac{5}{5} \ or \ 1$
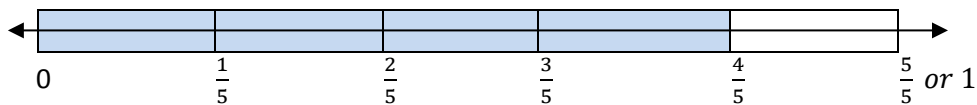
4 copies of the 5 equal parts represent the fractional amount

As students counted with whole numbers, they should also count with fractions. Counting equal-sized parts helps students determine the number of parts it takes to make a whole and recognize fractions that are equivalent to whole numbers.

Tools / Resources
Illustrative Mathematics Tasks:
3.NF Naming the Whole for a Fraction
3.MD, 3.G, 3.NF Halves, thirds, and sixths

See: "Fraction Reactions," NCSM, Great Tasks for Mathematics K-5, (2013).

See: For detailed explanations and examples of Standards

Common Misconceptions:
The idea that the smaller the denominator, the smaller the piece or part of the set, or the larger the denominator, the larger the piece or part of the set, is based on the comparison that in whole numbers, the smaller a number, the less it is, or the larger a number, the more it is. The use of different models, such as fraction bars and number lines, allows students to compare unit fractions to reason about their sizes.

Students think all shapes can be divided the same way. Present shapes other than circles, squares or rectangles to prevent students from over generalizing that all shapes can be divided the same way. For example, have students fold a triangle into eighths. Provide oral directions for folding the triangle:

1.   Fold the triangle into half by folding the left vertex (at the base of the triangle) over to meet the right vertex.
2.   Fold in this manner two more times.
3.   Have students label each eighth using fractional notation. Then, have students count the fractional parts in the triangle (one-eighth, two-eighths, three-eighths, and so on).

Student count "tick marks" on number lines (linear/length model) rather than the distance or region partitioned.

Domain: Number and Operations—Fractions (NF)

Standard: Grade 3.NF.2
Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a. Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line.

b. Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:
See Grade 3.NF.1

Explanation and Examples:
Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

The number line diagram is the first time students work with a number line for numbers that are between whole numbers (e.g., that $\frac{1}{2}$ is between 0 and 1).

In the number line diagram below, the space between 0 and 1 is divided (partitioned) into 4 equal regions. The distance from 0 to the first segment is 1 of the 4 segments from 0 to 1 or $\frac{1}{4}$ (**3.NF.2a**). Similarly, the distance from 0 to the third segment is 3 segments that are each one-fourth long. Therefore, the distance of 3 segments from 0 is the fraction $\frac{3}{4}$ (**3.NF.2b**).
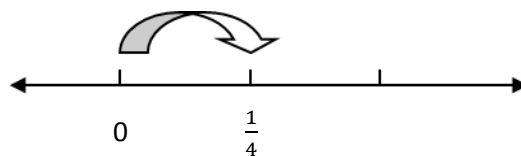
Students transfer their understanding of parts of a whole to partition a number line into equal parts. There are two new concepts addressed in this standard which students should have time to develop.

1. On a number line from 0 to 1, students can partition (divide) it into equal parts and recognize that each segmented part represents the same length.
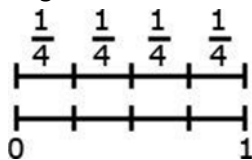
2. Students label each fractional part based on how far it is from zero to the endpoint.
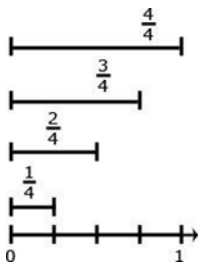
An interactive whiteboard may be used to help students develop these concepts.

Instructional Strategies:
See Grade 3.NF.1

Tools / Resources:
See: "Fraction Reactions," NCSM, Great Tasks for Mathematics K-5, (2013).

Illustrative Mathematics Tasks:
3.NF Locating Fractions Less than One on the Number Line
3.NF Closest to 1/2
3.NF Locating Fractions Greater than One on the Number Line
3.NF Find 1
3.NF Find 2/3
3.NF Which is Closer to 1?
3.NF Find 1/4 Starting from 1, Assessment Version
3.NF Find 7/4 starting from 1, Assessment Variation
3.NF Find 1 Starting from 5/3, Assessment Variation

Common Misconceptions:
See Grade 3.NF.1

Domain: Number and Operations—Fractions (NF)

Cluster: _Develop and understanding of fractions as numbers._

Standard: Grade 3.NF.3

Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b. Recognize and generate simple equivalent fractions, (e.g., $\frac{1}{2} = \frac{2}{4}, \frac{4}{6} = \frac{2}{3}$). Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. _Examples: Express 3 in the form $3 = \frac{3}{1}$; recognize that $\frac{6}{1} = 6$; locate $\frac{4}{4}$ and 1 at the same point of a number line diagram._

d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:
See Grade 3.NF.1

Explanation and Examples:

Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

An important concept when comparing fractions is to look at the size of the parts and the number of the parts. For example, $\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when 1 whole is cut into 2 pieces. (Students can SEE this by modeling----folding paper in half, in half again, and so on.

3.NF.3a and **3.NF.3b** Call for students to use visual fraction models (area models) and number lines to explore the idea of equivalent fractions. Students should only explore equivalent fractions using models, rather than using algorithms or procedures.

This standard includes writing whole numbers as fractions. The concept relates to fractions as division problems, where the fraction $\frac{3}{1}$ is 3 wholes divided into one group. This standard is the building block for later work where students divide a set of objects into a specific number of groups. Students must understand the meaning of $\frac{a}{1}$.

Major   Supporting   Additional   Depth Opportunities(DO)

Example:

If 6 brownies are shared between 2 people, how many brownies would each person get?

This standard involves comparing fractions with or without visual fraction models including number lines. Experiences should encourage students to reason about the size of pieces, the fact that $\frac{1}{3}$ of a cake is larger than $\frac{1}{4}$ of the same cake. Since the same cake (the whole) is split into equal pieces, thirds are larger than fourths.

In this standard, students should also reason that comparisons are only valid if the wholes are identical. For example, $\frac{1}{2}$ of a large pizza is a different amount than $\frac{1}{2}$ of a small pizza. Students should be given opportunities to discuss and reason about which $\frac{1}{2}$ is larger.

An important concept when comparing fractions is to look at the size of the parts and the number of the parts. For example, $\frac{1}{8}$ is smaller than $\frac{1}{2}$ because when 1 whole is cut into 8 pieces, the pieces are much smaller than when 1 whole is cut into 2 pieces.

Students recognize when examining fractions with common denominators, the wholes have been divided into the same number of equal parts. So the fraction with the larger numerator has the larger number of equal parts.

$$\frac{2}{6} < \frac{5}{6}$$

To compare fractions that have the same numerator but different denominators, students understand that each fraction has the same number of equal parts but the size of the parts are different. They can infer that the same number of smaller pieces is less than the same number of bigger pieces.

$$\frac{3}{8} < \frac{3}{4}$$

Instructional Strategies:
See Grade 3.NF.1

Tools/Resources:

Illustrative Mathematics Tasks:

3.NF Ordering Fractions

3.NF Comparing Fractions

3.NF Snow Day

3.NF Jon and Charlie's Run

3.MD, 3.G, 3.NF Halves, thirds, and sixths

3.NF Comparing Fractions with a Different Whole

3.NF Comparing Fractions with the Same Denominator, Assessment Variation

3.NF Comparing Fractions with the Same Numerators, Assessment Variation

3.NF Fraction Comparisons With Pictures, Assessment Variation

"Making a Cake", Georgia Department of Education.

See: Grade 3-5 Number and Operations Fractions Learning Progressions for detailed information:

Common Misconceptions:

See Grade 3.NF.1

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Measurement and Data (MD)

Standard: Grade 3.MD.1
Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: (3.MD.1-2)
This cluster is related to:
- Third Grade Critical Areas of Focus, **Solving multi-step problems**.
- Work with time and money. Grade 2 MD 7

Explanation and Examples:
This standard calls for students to solve elapsed time, including word problems. Students could use clock models or number lines to solve. On the number line, students should be given the opportunities to determine the intervals and size of jumps on their number line. Students could use pre-determined number lines (intervals every 5 or 15 minutes) or open number lines (intervals determined by students).

Students in second grade learned to tell time to the nearest five minutes. In third grade, they extend telling time and measure elapsed time both in and out of context using clocks and number lines.

Students may use a whiteboard to demonstrate understanding and justify their thinking.

Instructional Strategies: (3.MD.1-2)
A clock is a common instrument for measuring time. Learning to tell time has much to do with learning to read a dial-type instrument rather than with time measurement.

Students have experience in telling and writing time from analog and digital clocks to the hour and half hour in Grade 1 and to the nearest five minutes, using a.m. and p.m. in Grade 2. Now students will tell and write time to the nearest minute and measure time intervals in minutes.

Provide geared analog clocks that allow students to move the minute hand.

Students need experience representing time from a digital clock to an analog clock and vice versa.

Major   Supporting   Additional   Depth Opportunities(DO)

Provide word problems involving addition and subtraction of time intervals in minutes. Have students represent the problem on a number line. Student should relate using the number line with subtraction from Grade 2.

Provide opportunities for students to use appropriate tools to measure and estimate liquid volumes in liters only and masses of objects in grams and kilograms. Students need practice in reading the scales on measuring tools since the markings may not always be in intervals of one. The scales may be marked in intervals of two, five or ten.

Allow students to hold gram and kilogram weights in their hand to use as a benchmark. Use water colored with food coloring so that the water can be seen in a beaker.

Students should estimate volumes and masses before actually finding the measuring. Show students a group containing the same kind of objects. Then, show them one of the objects and tell them its weight. Fill a container with more objects and ask students to estimate the weight of the objects.

Use similar strategies with liquid measures. Be sure that students have opportunities to pour liquids into different size containers to see how much liquid will be in certain whole liters. Show students containers and ask, "How many liters do you think will fill the container?"

If making several estimates, students should make an estimate, then the measurement and continue the process of estimating measure rather than all estimates and then all measures. It is important to provide feedback to students on their estimates by using measurement as a way of gaining feedback on estimates.

Resources/Tools:
For detailed information see Measurement Learning Progression

See EngageNY Modules

Common Misconceptions:
Students may read the mark on a scale that is below a designated number on the scale as if it was the next number. For example, a mark that is one mark below 80 grams may be read as 81 grams. Students realize it is one away from 80, but do not think of it as 79 grams.

Avoid the use of paper plate clocks. Students need to see the actual relationship between the hour and the minute hand. This is not adequately represented on student made clocks.

Students forget to label the measurement or choose the incorrect unit.

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Measurement and Data (MD)

Cluster: *Solve problems involving measurement and estimation interval of time, liquid volume, and masses of objects.*

Standard: Grade 3. MD.2

Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). (Excludes compound units such as cm^3 and finding the geometric volume of a container.) Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:
See Grade 3.MD.1

Explanation and Examples:
This standard excludes multiplicative comparison problems (problems involving notions of "times as much".

This standard asks for students to reason about the units of mass and volume. Students need multiple opportunities weighing classroom objects and filling containers to help them develop a basic understanding of the size and weight of a liter, a gram, and a kilogram. Milliliters may also be used to show amounts that are less than a liter. Word problems should only be one-step and include the same units.

Example:
Students identify 5 things that have a mass of about one gram. They record their findings with words and pictures. (Students can repeat this for 5 grams and 10 grams.) This activity helps develop gram benchmarks. One large paperclip weighs about one gram. A box of large paperclips (100 clips) has a mass of about 100 grams so 10 boxes would have a mass of one kilogram.

Example:
A paper clip has a mass of about a) a gram, b) 10 grams, c) 100 grams?

Foundational understandings to help with measure concepts:
- Understand that larger units can be subdivided into equivalent units (partition).
- Understand that the same unit can be repeated to determine the measure (iteration).
- Understand the relationship between the size of a unit and the number of units needed (compensatory principal).

Major   Supporting   Additional   Depth Opportunities(DO)

Instructional Strategies:
See 3.MD.2

Students need multiple opportunities "massing" classroom objects and filling containers to help them develop a basic understanding of the size and mass of a liter, a gram, and a kilogram. Milliliters may also be used to show amounts that are less than a liter.

Tools/Resources:
See: K-5 Measurement See Learning Progressions for detailed information

Illustrative Math site task:
3.MD How Heavy?

Common Misconceptions:
Students often focus on size to determine estimates of mass. They can be confused by a big fluffy object and a tiny dense object. Because students cannot tell actual mass until they have handled an object, it is important that teachers do not ask students to estimate the mass of objects until they have had the opportunity to lift the objects and then make an estimate of the mass.

Domain: Measurement and Data (MD)

Standard: Grade 3. MD. 3

Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: (3.MD.3-4)

This cluster is connected to:

- Third Grade Critical Areas of Focus #2, **Developing understanding of fractions, especially unit fractions (fractions with numerator 1)** and goes beyond to address **Solving multi-step problems**.
- Represent and solve problems involving multiplication and division. (Grade 3 OA 1 – 4)
- Multiply and divide within 100. (Grade 3 OA 7)
- Solve problems involving the four operations, and identify and explain patterns in arithmetic. (Grade 3 OA 8 – 9)
- Represent and interpret data. (Grade 2 MD 9 – 10)

Explanation and Examples:

Students should have opportunities reading and solving problems using scaled graphs before being asked to draw one. Graphs on the next page all use five as the scale interval, but students should experience different intervals to further develop their understanding of scale graphs and number facts.

While exploring data concepts, students should 1) **Pose** a question, 2) **Collect** data, 3) **Analyze** data, and 4) **Interpret** data *(PCAI).* Students should be graphing data that is relevant to their lives.

Example:

Pose a question: What are some of the questions that could be asked of the date we see? Students should come up with a question. What is the typical genre read in our class? Collect and organize data: student survey.

Pictographs: Scaled pictographs include symbols that represent multiple units. Below is an example of a pictograph with symbols that represent multiple units. Graphs should include a title, scale, categories, category label, and data. Students need to use both horizontal and vertical bar graphs.

Major   Supporting   Additional   Depth Opportunities(DO)

- If you were to purchase a book for the class library which would be the best genre? Why?

Example of Scaled Graph:

Number of Books Read	
Nancy	✧ ✧ ✧ ✧ ✧
Juan	✧ ✧ ✧ ✧ ✧ ✧ ✧
✧ = 5 Books	

- Single Bar Graphs: Students use both horizontal and vertical bar graphs. Bar graphs include a title, scale, scale label, categories, category label, and data.
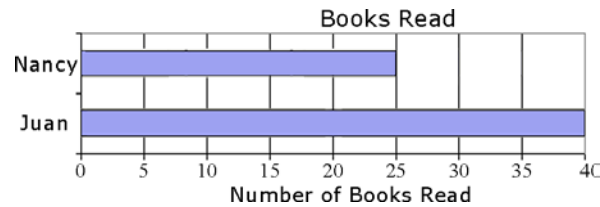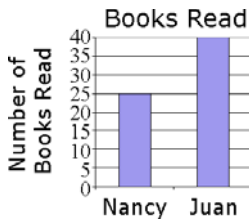
Analyze and Interpret data which could include:
- How many more nonfiction books where read than fantasy books?
- Did more people read biography and mystery books or fiction and fantasy books?
- About how many books in all genres were read?
- Using the data from the graphs, what type of book was read more often than a mystery but less often than a fairytale?
- What interval was used for this scale?
- What can we say about types of books read? What is a typical type of book read?
- If you were to purchase a book for the class library which would be the best genre?

Instructional Strategies: **(3.MD.3-4)**
Representation of a data set is extended from picture graphs and bar graphs with single-unit scales to scaled picture graphs and scaled bar graphs. Intervals for the graphs should relate to multiplication and division with 100 (product is 100 or less and numbers used in division are 100 or less).

In picture graphs, use values for the icons in which students are having difficulty with multiplication facts. For example, ☐ represents 7 people. If there are three ☐, stude 21 people. The intervals on the vertical scale in bar graphs should not exceed 100.

Students are to draw picture graphs in which a symbol or picture represents more than one object). Bar graphs are drawn with intervals greater than one. Ask questions that require students to compare quantities and use mathematical concepts and skills. Use symbols on picture graphs that student can easily represent half of, or know how many half of the symbol represents.

Students are to measure lengths using rulers marked with halves and fourths of an inch and record the data on a line plot. The horizontal scale of the line plot is marked off in whole numbers, halves or fourths. Students can create rulers with appropriate markings and use the ruler to create the line plots.

Resources/Tools

"Bar Grapher", NCTM.org, Illuminations. This is a NCTM site that contains a bar graph tool to create bar graphs.

"It's All About Multiplication-Exploring Equal Sets", NCTM.org, Illuminations.
Students listen to the counting story, *What Comes in 2's, 3's, & 4's*, and then use counters to set up multiple sets of equal size. They fill in a table listing the number of sets, the number of objects in each set, and the total number in all. They study the table to find examples of the order (commutative) property. Finally, they apply the equal sets model of multiplication by creating pictographs in which each icon represents several data points.

See: "What's in a Name? – Creating Pictographs", NCTM, Illuminations. This is a series of lesson in which student use data tools, one of which is pictographs answer questions about the data set.

"Barnyard Legs", Georgia Department of Education. Students solve multiplication problems using different strategies based on Amanda Bean's Amazing Dream, A Mathematical Story by Cindy Neuschwander or a similar book about multiplication.

"Guess Who's Coming to Dinner", Georgia Department of Education. Students are to arrange 18 people at 6 different card tables. Each table must be full and there must be an adult at each table. Students will use perimeter to find the solution.

Illustrative Mathematics Task:
3.OA, MD, NBT Classroom Supplies

Common Misconceptions:
Although intervals on a bar graph are not in single units, students count each square as one. To avoid this error, have students include tick marks between each interval. Students should begin each scale with 0. They should think of skip-counting when determining the value of a bar since the scale is not in single units.

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Measurement and Data (MD)

Cluster: Represent and interpret data.

Standard: Grade 3.MD.4
Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units— whole numbers, halves, or quarters.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:
See Grade 3.MD.3

Explanation and Examples:
Students in second grade measured length in whole units using both metric and U.S. customary systems. It's important to review with students how to read and use a standard ruler including details about halves and quarter marks on the ruler. Students should connect their understanding of fractions to measuring to one-half and one-quarter inch. Third graders need many opportunities measuring the length of various objects in their environment.

This standard provides a context for students to work with fractions by measuring objects to a quarter of an inch.

Example:

Measure objects in your desk to the nearest $\frac{1}{2}$ or $\frac{1}{4}$ of an inch, display data collected on a line plot.   How many objects measured $\frac{1}{4}$? $\frac{1}{2}$? etc.

Some important ideas related to measuring with a ruler are:

- The starting point of where one places a ruler to begin measuring
- Measuring is approximate. Items that students measure will not always measure exactly $\frac{1}{4}, \frac{1}{2}$ or one whole inch. Students will need to decide on an appropriate estimate length.
- Making paper rulers and folding to find the half and quarter marks will help students develop a stronger understanding of measuring length

Students generate data by measuring and create a line plot to display their findings. An example of a line plot is shown below:

Number of Objects Measured
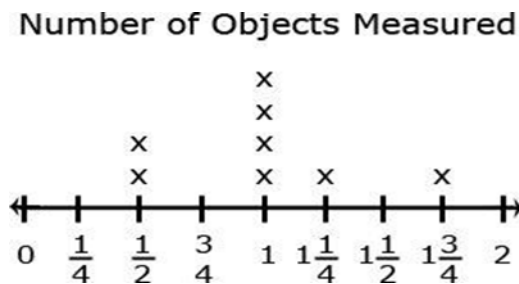
Instructional Strategies:
See Grade 3.MD.3

Tools/Resources:
See EngageNY Modules

Common Misconceptions:
See Grade 3.MD.3

Domain: Measurement an Data (MD)

Standard: Grade 3.MD.5

Recognize area as an attribute of plane figures and understand concepts of area measurement.

    a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

    b. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.

Suggested Standards for Mathematical Practice (MP):

✓ MP.2 Reason abstractly and quantitatively.

✓ MP.4 Model with mathematics.

✓ MP.5 Use appropriate tools strategically.

✓ MP.6 Attend to precision.

Connections: (3.MD.5-7)

This cluster is connected to:

- Third Grade Critical Area of Focus #3, **Developing understanding of the structure of rectangular arrays and of area.**
- Fluently multiply and divide within 100 (3.OA.2.7).
- Distributive property

Explanation and Examples: (3.MD.5-7)

These standards call for students to explore the concept of covering a region with "unit squares," which could include square tiles or shading on grid or graph paper.

Students develop understanding of using square units to measure area by:

- Using different sized square units
- Filling in an area with the same sized square units and counting the number of square units
- An interactive whiteboard would allow students to see that square units can be used to cover a plane figure.
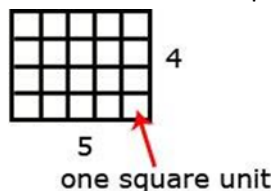
Instructional Strategies: **(3.MD.5-7)**

Students can cover rectangular shapes with tiles and count the number of units (tiles) to begin developing the idea that area is a measure of covering. Area describes the size of an object that is two-dimensional. The formulas should not be introduced before students discover the meaning of area.

The area of a rectangle can be determined by having students lay out unit squares and count how many square units it takes to completely cover the rectangle completely without overlaps or gaps.

Students need to develop the meaning for computing the area of a rectangle. A connection needs to be made between the number of squares it takes to cover the rectangle and the dimensions of the rectangle. Ask questions such as:

- What does the length of a rectangle describe about the squares covering it?
- What does the width of a rectangle describe about the squares covering it?

Tools/Resources:
Illustrative Math Task:
3.MD The Square Counting Shortcut

Common Misconceptions:
Students may confuse perimeter and area when they measure the sides of a rectangle and then multiply. They think the attribute they find is length, which is perimeter. Pose problems situations that require students to explain whether they are to find the perimeter or area.

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Measurement and Data (MD)

Standard: Grade 3.MD.6

Measure area by counting unit squares (square cm, square m, square in., square ft. and improvised units.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections:
See Grade 3.MD.5

Explanation and Examples:

Students should be counting the square units to find the area could be done in metric, customary, or non-standard square units. Using different sized graph paper, students can explore the areas measured in square centimeters and square inches.

An interactive whiteboard may also be used to display and count the unit squares (area) of a figure.

Tools/Resources:
Illustrative Math Tasks:

[3.MD, 3.G, 3.NF Halves, thirds, and sixths](#)
[3.MD Finding the Area of Polygons](#)

Common Misconceptions:
See Grade 3.MD.5

Domain: Measurement and Data (MD)

Standard: Grade 3.MD.7

Relate area to the operations of multiplication and addition.

   a.   Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b.   Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c.   Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   d.   Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Suggested Standards for Mathematical Practice (MP):

   ✓   MP.1   Make sense of problems and persevere in solving them.
   ✓   MP.2   Reason abstractly and quantitatively.
   ✓   MP.4   Model with mathematics.
   ✓   MP.5   Use appropriate tools strategically.
   ✓   MP.6   Attend to precision.

Connections:

See Grade 3.MD.5

Explanation and Examples:

Students should tile rectangle then multiply the side lengths to show it is the same.

To find the area one could count the squares or multiply 3 x 4 = 12.

1	2	3	4
5	6	7	8
9	10	11	12

Students should solve real world and mathematical problems.

Example:

Drew wants to tile the bathroom floor using 1 foot tiles. How many square foot tiles will he need?

6 ft.

8 ft.

1 square unit = ☐
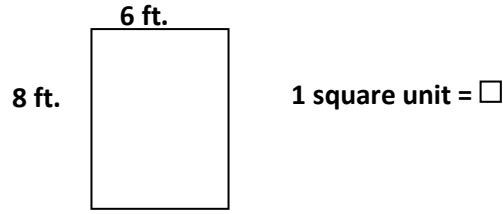
This standard extends students' work with the **distributive property**. For example, in the picture below the area of a 7 x 6 figure can be determined by finding the area of a 5 x 6 and 2 x 6 and adding the two sums.

5x6

2x6
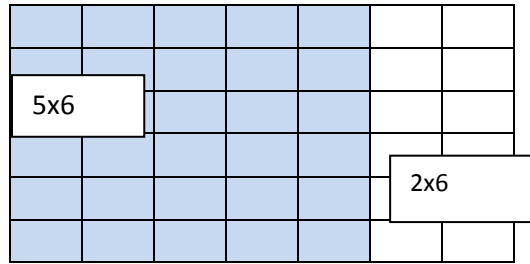
Students tile areas of rectangles, determine the area, record the length and width of the rectangle, investigate the patterns in the numbers, and discover that the area is the length times the width.

Example:

Joe and John made a poster that was 4ft. by 3ft. Melisa and Debbie made a poster that was 4ft. by 2ft. They placed their posters on the wall side-by-side so that that there was no space between them. How much area will the two posters cover?

Students use pictures, words, and numbers to explain their understanding of the distributive property in this context.

4′

3′   2′

a

a x b   a x c

b   c

4 x 3 + 4 x 2 = 20

4 (3 + 2) = 20

4 x 5 = 20
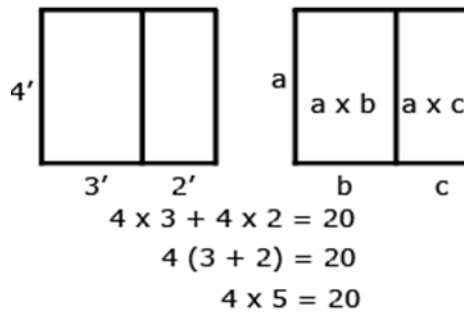
Example:

Students can decompose a rectilinear figure into different rectangles. They find the area of the figure by adding the areas of each of the rectangles together.
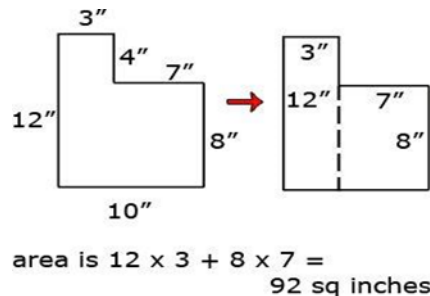
3″

4″

7″

12″

8″

10″

3″

12″   7″

8″

area is 12 x 3 + 8 x 7 =

92 sq inches

Instructional Strategies:
See Grade 3.MD.5

Tools/Resources:
See: for detailed information in:
[Learning Progressions-Measurement and Data (measurement part)](#)
[Learning Progressions-Measurement and Data (data part)](#)

Illustrative Math Tasks:
[3.MD Finding the Area of Polygons](#)
[3.MD Three Hidden Rectangles](#)

See: "Playful Puppies", NCSM, Great Tasks for Mathematics K-5, (2013)

Common Misconceptions:
See Grade 3.MD.5

Domain: Measurement and Data (MD)

Standard: Grade 3. MD.8
Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of other.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections:
This cluster is connected to:
- Third Grade Critical Area of Focus #3, **Developing understanding of the structure of rectangular arrays and of area**.
- Measure and estimate lengths in standard units. Grade 2 MD 1 – 4
- Relate addition and subtraction to length. Grade 2 MD 5 – 6

Explanation and Examples:
Students develop an understanding of the concept of perimeter by walking around the perimeter of a room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects; use addition to find perimeters; and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles.

Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given perimeter (e.g., find the rectangles with a perimeter of 14 cm.) They record all the possibilities using dot or graph paper, compile the possibilities into an organized list or a table, and determine whether they have all the possible rectangles.

Given a perimeter and a length or width, students use objects or pictures to find the missing length or width. They justify and communicate their solutions using words, diagrams, pictures, numbers, and an interactive whiteboard.

Major   Supporting   Additional   Depth Opportunities(DO)

Students use geoboards, tiles, graph paper, or technology to find all the possible rectangles with a given area (e.g. find the rectangles that have an area of 12 square units.) They record all the possibilities using dot or graph paper, compile the possibilities into an organized list or a table, and determine whether they have all the possible rectangles. Students then investigate the perimeter of the rectangles with an area of 12.

Area	Length	Width	Perimeter
12 sq. in.	1 in.	12 in.	26 in.
12 sq. in.	2 in.	6 in.	16 in.
12 sq. in	3 in.	4 in.	14 in.
12 sq. in	4 in.	3 in.	14 in.
12 sq. in	6 in.	2 in.	16 in.
12 sq. in	12 in.	1 in.	26 in.

The patterns in the chart allow the students to identify the factors of 12, connect the results to the commutative property, and discuss the differences in perimeter within the same area. This chart can also be used to investigate rectangles with the same perimeter. It is important to include squares in the investigation.

Instructional Strategies:
Students have created rectangles when they were finding the area of rectangles and connecting them to using arrays in the multiplication of whole numbers.

To explore finding the perimeter of a rectangle, have students use non-stretchy string.
- They should measure the string and create a rectangle before cutting it into four pieces.
- Have the student make four pieces so that there are two pieces of one length and two pieces of a longer or shorter length.
- Students should be able to make the connection that perimeter is the total distance around the rectangle.

Geoboards can be used to find the perimeter of rectangles also. Provide students with different perimeters and have them create the rectangles on the geoboards. Have students share their rectangles with the class. Have discussions about how different rectangles can have the same perimeter with different side lengths.

Students experienced measuring lengths of inches and centimeters in Grade 2. They have also related addition to length and writing equations with a symbol for the unknown to represent a problem.

- Once students know how to find the perimeter of a rectangle, they can find the perimeter of rectangular-shaped objects in their environment.
- They can use appropriate measuring tools to find lengths of rectangular-shaped objects in the classroom.
- Present problems situations involving perimeter, such as finding the amount of fencing needed to enclose a rectangular shaped park, or how much ribbon is needed to decorate the edges of a picture frame.
- Present problem situations in which the perimeter and two or three of the side lengths are known, requiring students to find the unknown side length.

Students need to recognize when a problem situation requires a solution relates to the perimeter or the area.

Major   Supporting   Additional   Depth Opportunities(DO)

They should have experience with understanding area concepts when they recognize it as an attribute of plane figures. They also discovered that when plane figures are covered without gaps by *n* unit squares, the area of the figure is *n* square units.

Students need to explore how measurements are affected when one attribute to be measured is held constant and the other is changed. Using square tiles, students can discover that the area of rectangles may be the same, but the perimeter of the rectangles varies. Geoboards can also be used to explore this same concept.

Resources/Tools:
Illustrative Mathematics Tasks:
3.MD Shapes and their Insides

Common Misconceptions:
Students think that when they are presented with a drawing of a rectangle with only two of the side lengths shown or a problem situation with only two of the side lengths provided, these are the only dimensions they should add to find the perimeter. Encourage students to include the appropriate dimensions on the other sides of the rectangle. With problem situations, encourage students to make a drawing to represent the situation in order to find the perimeter.

Domain: Geometry (G)

Standard: Grade 3.G.1
Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:
This cluster is connected to:
- Third Grade Critical Areas of Focus #3, **Developing understanding of the structure of rectangular arrays and of area** and #4, **Describing and analyzing two-dimensional shapes.**
- Reason with shapes and their attributes. (Grade 2 G 3)

Explanation and Examples:
In second grade, students identify and draw triangles, quadrilaterals, pentagons, and hexagons. Third graders build on this experience and further investigate quadrilaterals (technology may be used during this exploration).

Students recognize shapes that are and are not quadrilaterals by examining the properties of the geometric figures.

They conceptualize that a quadrilateral must be a closed figure with four straight sides and begin to notice characteristics of the angles and the relationship between opposite sides. Students should be encouraged to provide details and use proper vocabulary when describing the properties of quadrilaterals. They sort geometric figures (see examples below) and identify squares, rectangles, and rhombuses as quadrilaterals.
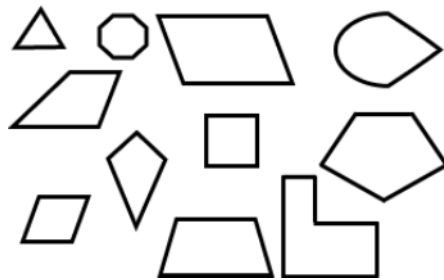
Students should classify shapes by attributes and drawing shapes that fit specific categories.

For example, parallelograms include: squares, rectangles, rhombi, or other shapes that have two pairs of parallel sides. Also, the broad category quadrilaterals include all types of parallelograms, trapezoids and other four-sided figures.

Major   Supporting   Additional   Depth Opportunities(DO)

Example:
Draw a picture of a quadrilateral. Draw a picture of a rhombus.

How are they alike? How are they different?

Is a quadrilateral a rhombus? Is a rhombus a quadrilateral? Justify your thinking.

Instructional Strategies: (3.G.1-2)

In earlier grades, students have experiences with informal reasoning about particular shapes through sorting and classifying using their geometric attributes. Students have built and drawn shapes given the number of faces, number of angles and number of sides.

The focus now is on identifying and describing properties of two-dimensional shapes in more precise ways using properties that are shared rather than the appearances of individual shapes. These properties allow for generalizations of all shapes that fit a particular classification.

Development in focusing on the identification and description of shapes' properties should include examples and non-examples, as well as examples and non-examples drawn by students of shapes in a particular category.

For example, students could start with identifying shapes with right angles. An explanation as to why the remaining shapes do not fit this category should be discussed. Students should determine common characteristics of the remaining shapes.
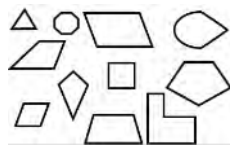
In Grade 2, students partitioned rectangles into two, three or four equal shares, recognizing that the equal shares need not have the same shape. They described the shares using words such as, halves, thirds, half of, a third of, etc., and described the whole as two halves, three thirds or four fourths.

In Grade 4, students will partition shapes into parts with equal areas (the spaces in the whole of the shape). These equal areas need to be expressed as unit fractions of the whole shape, i.e., describe each part of a shape partitioned into four parts as $\frac{1}{4}$ of the area of the shape.
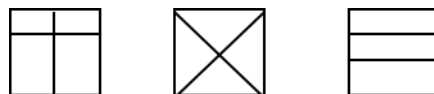
Have students draw different shapes and see how many ways they can partition the shapes into parts with equal area.

Resources/Tools:

See Geometry Learning Progressions for detailed information:

"3-D Detectives", Georgia Department of Education. Students identify, describe and illustrate plane and solid figures according to geometric properties.

"What's In A Name", Georgia Department of Education. Students describe and classify plane figures (triangles, square, rectangle, trapezoid, quadrilateral, pentagon, hexagon, and irregular polygonal shapes) by the number of edges, vertices and angles.

Common Misconceptions: (3.G.1-2)

Students may identify a square as a "non-rectangle" or a "non-rhombus" based on limited images they see. They do not recognize that a square is a rectangle because it has all of the properties of a rectangle. They may list properties of each shape separately, but not see the interrelationships between the shapes.

For example, students do not look at the properties of a square that are characteristic of other figures as well.

Using straws to make four congruent figures have students change the angles to see the relationships between a rhombus and a square. As students develop definitions for these shapes, relationships between the properties will be understood.

Major   Supporting   Additional   Depth Opportunities(DO)

Domain: Geometry (G)

Standard: Grade 3. G.2

Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as $\frac{1}{4}$ of the area of the shape.*

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2   Reason abstractly and quantitatively.
- ✓ MP.4   Model with mathematics.
- ✓ MP.5   Use appropriate tools strategically.

Connections:
See Grade 3.G.1

Explanation and Examples:

This standard builds on students' work with fractions and area. Students are responsible for partitioning shapes into halves, thirds, fourths, sixths and eighths.

Example:

These figures are partitioned/divided into four equal parts. Each part is $\frac{1}{4}$ of the total area of the figure.

Squares partitioned into fourths

These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).
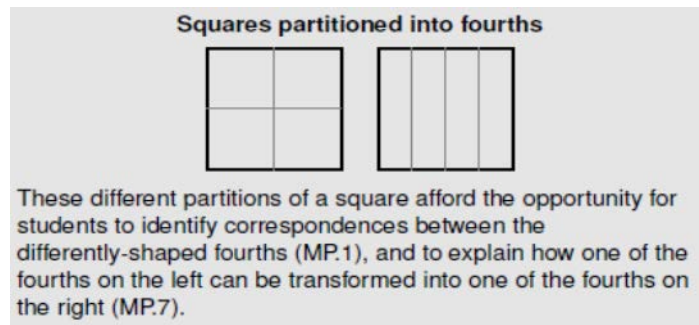
Examples:

This figure was partitioned/divided into four equal parts. Each part is $\frac{1}{4}$ of the total area of the figure.
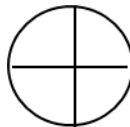
Major   Supporting   Additional   Depth Opportunities(DO)

Given a shape, students partition it into equal parts, recognizing that these parts all have the same area. They identify the fractional name of each part and are able to partition a shape into parts with equal areas in several different ways.

Given a shape, students partition it into equal parts, recognizing that these parts all have the same area. They identify the fractional name of each part and are able to partition a shape into parts with equal areas in several different ways.

Area representations of $\frac{1}{4}$

In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.
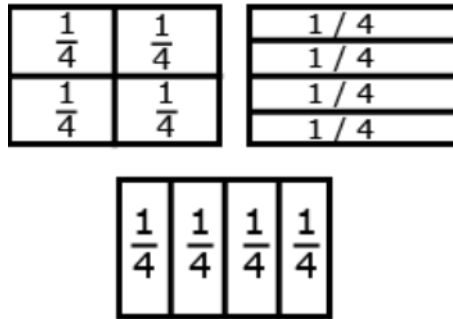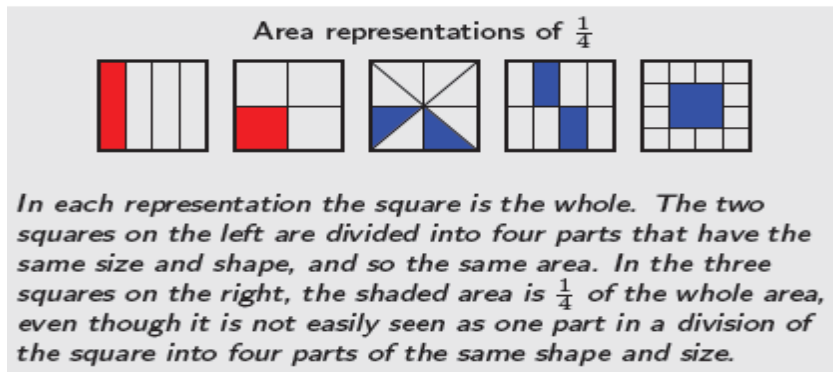
Learning Progressions:
Number & Operations-Fractions 3-5

Instructional Strategies:
See Grade 3.G.1

Tools/Resources:
Illustrative Math Tasks:
3.G Geometric pictures of one half
3.G Representing Half of a Circle
3.MD, 3.G, 3.NF Halves, thirds, and sixths

Common Misconceptions:
See Grade 3.G.1

Major   Supporting   Additional   Depth Opportunities(DO)

Appendix

TABLE 1. Common Addition and Subtraction Situations[6]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$

	Total Unknown	Addend Unknown	Both Addends Unknown[1]
Put Together / Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5$ or $5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5$ or $5 = 5 + 0$ $5 = 1 + 4$ or $5 = 4 + 1$ $5 = 2 + 3$ or $5 = 3 + 2$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5$ or $5 - 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?$ or $3 + 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?$ or $? + 3 = 5$

[1]These *take apart* situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.

[2]Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation especially for small numbers less than or equal to 10.

[3]For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

[6]Adapted from Box 2-4 of Mathematics Learning in Early Childhood, National Research Council (2009, pp. 32, 33).

TABLE 2. Common Multiplication and Division Situations[7]

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ And $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*: You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*: You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*: You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[4] Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*: What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*: A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*: A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*: A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*: A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example*: A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[4]The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

[5]Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

[7]The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

TABLE 3. The Properties of Operations

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every (a) there exists$(-a)$ so that $a + (-a) = (-a) + a = 0$
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $\frac{1}{a}$ so that $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

Here *a*, *b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

TABLE 4. The Properties of Equality*

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$ then $b = a$
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$
Addition property of equality	If $a = b$ then $a + c = b + c$
Subtraction property of equality	If $a = b$ then $a - c = b - c$
Multiplication property of equality	If $a = b$ then $a \times c = b \times c$
Division property of equality	If $a = b$ and $c \neq 0$ then $a \div c = b \div c$
Substitution property of equality	If $a = b$ then b may be substituted for a in any expression containing a.

Here *a*, *b* and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.
If $a > b$ and $b > c$ then $a > c$
If $a > b$ then $b < a$
If $a > b$ then $-a < -b$
If $a > b$ then $a \pm c > b \pm c$
If $a > b$ and $c > 0$ then $a \times c > b \times c$
If $a > b$ and $c < 0$ then $a \times c < b \times c$
If $a > b$ and $c > 0$ then $a \div c > b \div c$
If $a > b$ and $c < 0$ then $a \div c < b \div c$

Here *a*, *b* and *c* stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children

Levels	$8 + 6 = 14$	$14 - 8 = 6$
Level 1: Count all	**Count All** a: 1 2 3 4 5 6 7 8 (○○○○○ ○○○) 1 2 3 4 5 6 7 8 b: 1 2 3 4 5 6 (○○○○○ ○) 9 10 11 12 13 14	**Take Away** a: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 b: 1 2 3 4 5 6 7 8 1 2 3 4 5 6 (c)
Level 2: Count on	**Count On** 8 (○○○○○ ○○○) ○○○○○ ○ →8 9 10 11 12 13 14	To solve $14 - 8$ I count on $8 + ? = 14$ I took away 8 10 11 12 13 9 14 8 to 14 is 6 so $14 - 8 = 6$
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	**Recompose: Make a Ten** ○○○○○ ○○○ ○⊙○○○ ○ $10 + 4$ ○○○○○ ○○○ ○○○○○⊙ 10 — + — 4	$14 - 8$: I make a ten for $8 + ? = 14$ 8 + 2 + 4 6 8 + 6 = 14
Doubles $\pm n$	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) collection of items and understands that the last word tells "how many" even. Beyond on small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words

The Common Core State Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

References, Resources, and Links

1. Achieve the Core. http://www.achievethecore.org/.
2. Bolam, R., McMahon, A., Stoll, L., Thomas, S., & Wallace, M. (2005). Creating and sustaining professional learning communities. *Research Report Number 637*. General Teaching Council for England. London, England: Department for Education and Skills.
3. Croft, A., Coggshall, J. G., Dolan, M., Powers, E., with Killion, J. (2010). *Job-embedded professional development: What it is, who is responsible, and how to get it done well.* National Comprehensive Center for Teacher Quality. Retrieved March 11, 2013, from http://www.tqsource.org/
4. Darling-Hammond, L., Wei, R.C., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: a status report on teacher development in the United States and abroad.* Oxford, OH: National Staff Development Council and the School Redesign Network at Stanford University.
5. Garet, M. S., Porter, A. C., Desimone, L., Birman, B., & Yoon, K. (2001). What makes professional development effective? *American Educational Research Journal,* 38(4), 915–945. Retrieved March 11, 2013, from www.aztla.asu.edu/ProfDev1.pdf
6. Guskey, T. (2000). *Evaluating professional development.* Thousand Oaks, CA: Corwin Press.
7. Illustrative Mathematics. http://www.illustrativemathematics.org/.
8. Inside Mathematics. http://www.insidemathematics.org/.
9. Kanold, T. & Larson, M. (2012). *Common Core Mathematics in a PLC at Work, Leaders Guide.*
10. Kansas Mathematics Standards (CCSS). (2012). http://www.ksde.org/Default.aspx?tabid=4754
11. Killion, J. (2002). *Assessing impact: Evaluating staff development.* Oxford, OH: National Staff Development Council.
12. Learning Forward. (2011). *Standards for Professional Learning.*
13. Learn NC. http://www.learnnc.org/lp/editions/ccss2010-mathematics.
14. Learn Zillion. http://learnzillion.com/.
15. Mathematics Assessment Project. http://map.mathshell.org/materials/index.php.
16. McCallum, W., Daro, P., & Zimba, J. *Progressions Documents for the Common Core Math Standards.* Retrieved March 11, 2013, from *http://ime.math.arizona.edu/progressions/#about* .
17. Mid-continent Research for Education and Learning. (2000). *Principles in action: Stories of award-winning professional development* [video]. Aurora, CO: Author.
18. National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics.*
19. National Council of Teachers of Mathematics. (2000). *Principles & Standards for School Mathematics.*
20. National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence.*
21. National Council of Teachers of Mathematics. (2009) *Focus in High School Mathematics: Reasoning and Sense Making.*
22. National Council of Teachers of Mathematics. *Core Math Tools.* http://www.nctm.org/resources/content.aspx?id=32702.
23. National Council of Teachers of Mathematics. *Illuminations.* http://illuminations.nctm.org/.
24. National Mathematics Advisory Panel. (2008). *The Final Report of the National Mathematics Advisory Panel.* U.S. Department of Education.

25. National Staff Development Council. (2001). *Standards for staff development.* Oxford, OH: Author. Retrieved March 11, 2013, from http://www.nsdc.org/standards/index.cfm

26. Porter, A., Garet, M., Desimone, L., Yoon, K., & Birman, B. (2000). *Does professional development change teaching practice? Results from a three-year study.* Washington, DC: U.S. Department of Education. Retrieved March 11, 2013, from http://www.ed.gov/rschstat/eval/teaching/epdp/report.pdf

27. South Dakota Department of Education. *Curation Review Rubric for Mathematics Learning Resources.* Retrieved March 11, 2013, from http://www.livebinders.com/play/play?id=367846

28. Steiner, L. (2004). *Designing effective professional development experiences: What do we know?* Naperville, IL: Learning Point Associates.

29. Tools for the Common Core Standards. http://commoncoretools.me/.

30. Wisconsin Department of Public Instruction. (2010) *Characteristics of High-Quality and Evidence-Based Professional Learning*. Retrieved March 11, 2013 from, http://www.betterhighschools.org/MidwestSIG/documents/Rasmussen_Handout1.pdf

31. Yoon, K. S., Duncan T., Lee, S. W.-Y., Scarloss, B., & Shapley, K. L. (2007). *Reviewing the evidence on how teacher professional development affects student achievement*. (Issues & Answers Report, REL 2007–No. 033). Washington, DC: National Center for Education Evaluation and Regional Assistance, Regional Education Laboratory Southwest. Retrieved March 11, 2013, from http://ies.ed.gov/

32. Publishers Criteria: www.corestandards.org

33. Focus by Grade Level, Content Emphases by Jason Zimba: http://achievethecore.org/page/774/focus-by-grade-level

34. Georgie Frameworks: https://www.georgiastandards.org/Standards/Pages/BrowseStandards/MathStandards9-12.aspx

35. engageNY Modules: http://www.engageny.org/mathematicshttp://www.engageny.org/mathematics