Progressions for the Common Core State Standards in Mathematics (draft)

4 July 2013

Suggested citation:

Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The Number System; High School, Number.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Number System, 6–8

Overview

In Grades 6–8, students build on two important conceptions which have developed throughout K–5, in order to understand the rational numbers as a number system. The first is the representation of whole numbers and fractions as points on the number line, and the second is a firm understanding of the properties of operations on whole numbers and fractions.

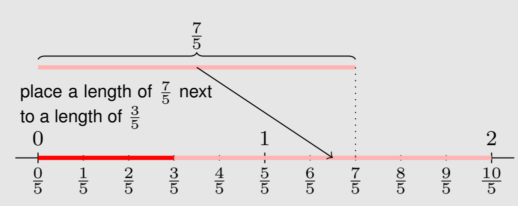
Representing numbers on the number line In early grades, students see whole numbers as counting numbers, Later, they also understand whole numbers as corresponding to points on the number line. Just as the 6 on a ruler measures 6 inches from the 0 mark, so the number 6 on the number line measures 6 units from the origin. Interpreting numbers as points on the number line brings fractions into the family as well; fractions are seen as measurements with new units, creating by partitioning the whole number unit into equal pieces. Just as on a ruler we might measure in tenths of an inch, on the number line we have halves, thirds, fifths, sevenths; the number line is a sort of ruler with every denominator. The denominators 10, 100, etc. play a special role, partioning the number line into tenths, hundredths, etc., just as a metric ruler is partioned into centimeters and millimeters.

Starting in Grade 2 students see addition as concatenation of lengths on the number line.[2.MD.6] By Grade 4 they are using the same model to represent the sum of fractions with the same denominator: $\frac{3}{5} + \frac{7}{5}$ is seen as putting together a length that is 3 units of one fifth long with a length that is 7 units of one fifth long, making 10 units of one fifths in all. Since there are five fifths in 1 (that's what it means to be a fifth), and 10 is 2 fives, we get $\frac{3}{5} + \frac{7}{5} = 2$. Two fractions with different denominators are added by representing them in terms of a common unit.

Representing sums as concatenated lengths on the number line is important because it gives students a way to think about addition that makes sense independently of how numbers are represented symbolically. Although addition calculations may look different for numbers represented in base ten and as fractions, addition is the

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Representing $\frac{3}{5}$ and $\frac{7}{5}$ on the number line



same operation in each case. Furthermore, the concatenation model of addition extends naturally to negative numbers in Grade 7.

Properties of operations The number line provides a representation that can be used to building understanding of sums and differences of rational numbers. However, building understanding of multiplication and division of rational numbers relies on a firm understanding of properties of operations. Although students have not necessarily been taught formal names for these properties, they have used them repeatedly in elementary school and have been with reasoning with them. The commutative and associative properties of addition and mutiplication have, in particular, been their constant friends in working with strategies for addition and multiplication.[1.OA.3, 3.OA.5]

The existence of the multiplicative identity (1) and multiplicative inverses start to play important roles as students learn about fractions. They might see fraction equivalence as confirming the identity rule for fractions. In Grade 4 they learn about fraction equivalence

$$\frac{n \times a}{n \times b} = \frac{a}{b}$$

and in Grade 5 they relate this to multiplication by 1

$$\frac{n \times a}{n \times b} = \frac{n}{n} \times \frac{a}{b} = 1 \times \frac{a}{b},$$

thus confirming that the identity rule

$$1 \times \frac{a}{b} = \frac{a}{b}$$

works for fractions.[5.NF.5]

As another example, the commutative property for multiplication plays an important role in understanding multiplication with fractions. For example, although

$$5 \times \frac{1}{2} = \frac{5}{2}$$

can be made sense of using previous understandings of whole number multiplication as repeated addition, the other way around,

$$\frac{1}{2} \times 5 = \frac{5}{2},$$

seems to come from a different source, from the meaning of phrases such as “half of” and a mysterious acceptance that “of” must mean multiplication. A more reasoned approach would be to observe that if we want the commutative property to continue to hold, then we must have

$$\frac{1}{2} \times 5 = 5 \times \frac{1}{2} = \frac{5}{2},$$

Properties of Operations on Rational Numbers

Properties of Addition

1. Commutative Property. For any two rational numbers a and b, $a + b = b + a$.
2. Associative Property. For any three rational numbers a, b and c, $(a + b) + c = a + (b + c)$.
3. Existence of Identity. The number 0 satisfies $0 + a = a = a + 0$.
4. Existence of Additive Inverse. For any rational number a, there is a number $-a$ such that $a + (-a) = 0$.

Properties of Multiplication

1. Commutative Property. For any two rational numbers a and b, $a \times b = b \times a$.
2. Associative Property. For any three rational numbers a, b and c, $(a \times b) \times c = a \times (b \times c)$.
3. Existence of Identity. The number 1 satisfies $1 \times a = a = a \times 1$.
4. Existence of Multiplicative Inverse. For every non-zero rational number a, there is a rational number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$.

The Distributive Property

For rational numbers a, b and c, one has $a \times (b + c) = a \times b + a \times c$.

1.OA.3 Apply properties of operations as strategies to add and subtract.[1]

3.OA.5 Apply properties of operations as strategies to multiply and divide.[2]

5.NF.5 Interpret multiplication as scaling (resizing), by:

a . . .

b . . . and relating the principle of fraction equivalence $\frac{a}{b} = \frac{n \times a}{n \times b}$ to the effect of multiplying $\frac{a}{b}$ by 1.

and that $\frac{5}{2}$ is indeed "half of five," as we have understood in Grade 5.[5.NF.3]

When students extend their conception of multiplication to include negative rational numbers, the properties of operations become crucial. The rule that the product of negative numbers is positive, often seen as mysterious, is the result of extending the properties of operations (particularly the distributive property) to rational numbers.

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

Grade 6

As Grade 6 dawns, students have a firm understanding of place value and the properties of operations. On this foundation they are ready to start using the properties of operations as tools of exploration, deploying them confidently to build new understandings of operations with fractions and negative numbers. They are also ready to complete their growing fluency with algorithms for the four operations.

Apply and extend previous understandings of multiplication and division to divide fractions by fractions In Grade 6 students conclude the work with operations on fractions, started in Grade 4, by computing quotients of fractions.[6.NS.1] In Grade 5 students divided unit fractions by whole numbers and whole numbers by unit fractions, two special cases of fraction division that are relatively easy to conceptualize and visualize.[5.NF.7ab] Dividing a whole number by a unit fraction can be conceptualized in terms of the measurement interpretation of division, which conceptualizes $a \div b$ as the the measure of of a by units of length b on the number line, that is, the solution to the multiplication equation $a = ? \times b$. Dividing a unit fraction by a whole number can be interpreted in terms of the sharing interpretation of division, which conceptualizes $a \div b$ as the size of a share when a is divided into b equal shares, that is, the solution to the multiplication equation $a = b \times ?$.

Now in Grade 6 students develop a general understanding of fraction division. They can use story contexts and visual models to develop this understanding, but also begin to move towards using the relation between division and multiplication.

For example, they might use the measurement interpretation of division to see that $\frac{8}{3} \div \frac{2}{3} = 4$, because 4 is 4 is how many $\frac{2}{3}$ there are in $\frac{8}{3}$. At the same time they can see that this latter statement also says that $4 \times \frac{2}{3} = \frac{8}{3}$. This multiplication equation can be used to obtain the division equation directly, using the relation between multiplication and division.

Quotients of fractions that are whole number answers are particularly suited to the measurement interpretation of division. When this interpretation is used for quotients of fractions that are not whole numbers, it can be rephrased from "how many times does this go into that?" to "how much of this is in that?" For example,

$$\frac{2}{3} \div \frac{3}{4}$$

can be interpreted as how many $\frac{3}{4}$-cup servings are in $\frac{2}{3}$ of a cup of yogurt, or as how much of a $\frac{3}{4}$-cup serving is in $\frac{2}{3}$ of a cup of yogurt. Although linguistically different the two questions are mathematically the same. Both can be visualized as in the margin and expressed using a multiplication equation with an unknown for the

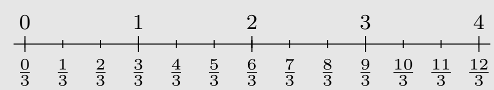
6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

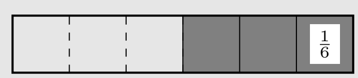
a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

Visual models for division of whole numbers by unit fractions and unit fractions by whole numbers

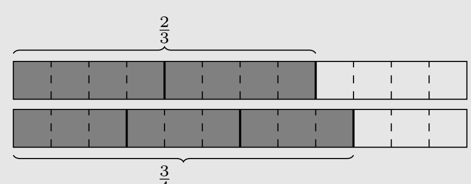


Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.



Reasoning with a fraction strip using the sharing interpretation of division: the strip is the whole and the shaded area is $\frac{1}{2}$ of the whole. If the shaded area is divided into 3 equal parts, then 2×3 of those parts make up the whole, so $\frac{1}{2} \div 3 = \frac{1}{2\times 3} = \frac{1}{6}$.

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{2}{3} = ? \times \frac{3}{4}$



We find a common unit for comparing $\frac{2}{3}$ and $\frac{3}{4}$ by dividing each $\frac{1}{3}$ into 4 parts and each $\frac{1}{4}$ into 3 parts. Then $\frac{2}{3}$ is 8 parts when $\frac{3}{4}$ is divided into 9 equal parts, so $\frac{2}{3} = \frac{8}{9} \times \frac{3}{4}$, which is the same as saying that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

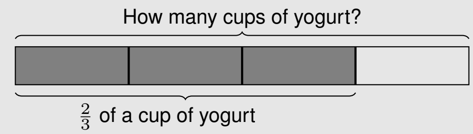
first factor:

$$\frac{2}{3} = ? \times \frac{3}{4}.$$

The same division problem can be interpreted using the sharing interpretation of division: how many cups are in a full container of yogurt when $\frac{2}{3}$ of a cup fills $\frac{3}{4}$ of the container. In other words, $\frac{3}{4}$ of what amount is equal to $\frac{2}{3}$ cups? In this case, $\frac{2}{3} \div \frac{3}{4}$ is seen as the solution to a multiplication equation with an unknown as the second factor:

$$\frac{3}{4} \times ? = \frac{2}{3}.$$

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{3}{4} \times ? = \frac{2}{3}$



The shaded area is $\frac{3}{4}$ of the entire strip. So it is 3 parts of a division of the strip into 4 equal parts. Another way of seeing this is that the strip is 4 parts of a division of the shaded area into 3 equal parts. That is, the strip is $\frac{4}{3}$ times the shaded part. So $? = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$.

There is a close connection between the reasoning shown in the margin and reasoning about ratios; if two quantities are in the ratio $3 : 4$, then there is a common unit so that the first quantity is 3 units and the second quantity is 4 units. The corresponding unit rate is $\frac{3}{4}$, and the first quantity is $\frac{3}{4}$ times the second quantity. Viewing the situation the other way around, with the roles of the two quantities interchanged, the same reasoning shows that the second quantity is $\frac{4}{3}$ times the first quantity. Notice that this leads us directly to the invert-and-multiply for fraction division: we have just reasoned that the ? in the equation above must be equal to $\frac{4}{3} \times \frac{2}{3}$, which is exactly what the rules gives us for $\frac{2}{3} \div \frac{4}{3}$.[6.NS.1]

The invert-and-multiply rule can also be understood algebraically as a consequence of the general rule for multiplication of fractions. If $\frac{a}{b} \div \frac{c}{d}$ is is defined to be the missing factor in the multiplication equation

$$? \times \frac{c}{d} = \frac{a}{b}$$

then the fraction that does the job is

$$? = \frac{ad}{bc},$$

as we can verify by putting it into the multiplication equation and using the already known rules of fraction multiplication and the properties of operations:

$$\frac{ad}{bc} \times \frac{c}{d} = \frac{(ad)c}{(bc)d} = \frac{a(cd)}{b(cd)} = \frac{a}{b} \times \frac{cd}{cd} = \frac{a}{b}.$$

6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

Compute fluently with multi-digit numbers and find common factors and multiples In Grade 6 students consolidate the work of earlier grades on operations with whole numbers and decimals by becoming fluent in the four operations on these numbers.[6.NS.2, 6.NS.3] Much of the foundation for this fluency has been laid in earlier grades. They have known since Grade 3 that whole numbers are fractions[3.NF.3c] and since Grade 4 that decimal notation is a way of writing fractions with denominator equal to a power of 10;[4.NF.6] by Grade 6 they start to see whole numbers, decimals and fractions

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

3.NF.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

not as wholly different types of numbers but as as part of the same number system, represented by the number line.

In many traditional treatments of fractions greatest common factors occur in reducing a fraction to lowest terms, and least common multiples occur in adding fractions. As explained in the Fractions Progression, neither of these activities is treated as a separate topic in the standards. Indeed, insisting that finding a least common multiple is an essential part of adding fractions can get in the way of understanding the operation, and the excursion into prime factorization and factor trees that is often entailed in these topics can be time-consuming and distract from the focus of K–5. In Grade 6, however, students experience a modest introduction to the concepts[6.NS.4] and put the idea of greatest common factor to use in a rehearsal for algebra, where they will need to see, for example, that $3x^2 + 6x = 3x(x + 2)$.

6.NS.4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

Apply and extend previous understandings of numbers to the system of rational numbers In Grade 6 the number line is extended to include negative numbers. Students initially encounter negative numbers in contexts where it is natural to describe both the magnitude of the quantity, e.g. vertical distance from sea level in meters, and the direction of the quantity (above or below sea level).[6.NS.5] In some cases 0 has an essential meaning, for example that you are at sea level; in other cases the choice of 0 is merely a convention, for example the temperature designated as 0° in Farenheit or Celsius. Although negative integers might be commonly used as initial examples of negative numbers, the Standards do not introduce the integers separately from the entire system of rational numbers, and examples of negative fractions or decimals can be included from the beginning.

Directed measurement scales for temperature and elevation provide a basis for understanding positive and negative numbers as having a positive or negative direction on the number line.[6.NS.6a] Previous understanding of numbers on the number line related the position of the number to measurement: the number 5 is located at the endpoint of an line segment 5 units long whose other endpoint is at 0. Now the line segments acquire direction; starting at 0 they can go in either the positive or the negative direction. These directed numbers can be represented by putting arrows at the endpoints of the line segments.

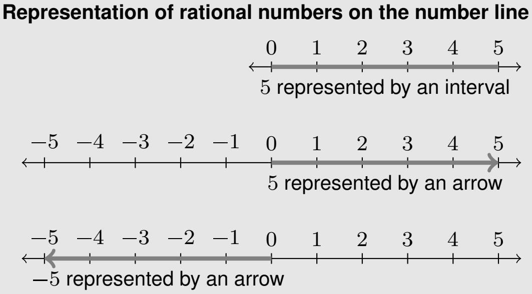
Students come to see $-p$ as the opposite of p, located an equal distance from 0 in the opposite direction. In order to avoid the common misconception later in algebra that any symbol with a negative sign in front of it should be a negative number, it is useful for students to see examples where $-p$ is a positive number, for example if $p = -3$ then $-p = -(-3) = 3$. Students come to see the operation of putting a negative sign in front of a number as flipping the direction of the number from positive to negative or negative to

6.NS.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

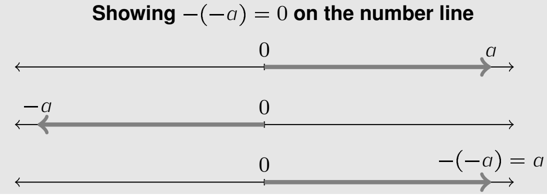
6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

a Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

Representation of rational numbers on the number line



Showing $-(-a) = 0$ on the number line



positive. Students generalize this understanding of the meaning of the negative sign to the coordinate plane, and can use it in locating numbers on the number line and ordered pairs in the coordinate plane.[6.NS.6bc]

With the introduction of negative numbers, students gain a new sense of ordering on the number line. Whereas statements like $5 < 7$ could be understood in terms of having more of or less of a certain quantity—"I have 5 apples and you have 7, so I have fewer than you"—comparing negative numbers requires closer attention to the relative positions of the numbers on the number line rather than their magnitudes.[6.MS.7a] Comparisons such as $-7 < -5$ can initially be confusing to students, because -7 is further away from 0 than -5, and is therefore larger in magnitude. Referring back to contexts in which negative numbers were introduced can be helpful: 7 meters below sea level is lower than 5 meters below sea level, and $-7°$ F is colder than $-5°$ F. Students are used to thinking of colder temperatures as lower than hotter temperatures, and so the mathematically correct statement also makes sense in terms of the context.[6.NS.7b]

At the same time, the prior notion of distance from 0 as a measure of size is still present in the notion of absolute value. To avoid confusion it can help to present students with contexts where it makes sense both to compare the order of two rational numbers and to compare their absolute value, and where these two comparisons run in different directions. For example, someone with a balance of \$100 in their bank account has more money than someone with a balance of $-\$1000$, because $100 > -1000$. But the second person's debt is much larger than the first person's credit $|-1000| > |100|$.[6.NS.7cd]

This understanding is reinforced by extension to the coordinate plane.[6.NS.8]

b Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

6.NS.7 Understand ordering and absolute value of rational numbers.

a Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram.

b Write, interpret, and explain statements of order for rational numbers in real-world contexts.

c Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.

d Distinguish comparisons of absolute value from statements about order.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Grade 7

Addition and subtraction of rational numbers In Grade 6 students learned to locate rational numbers on the number line; in Grade 7 they extend their understanding of operations with fractions to operations with rational numbers. Whereas previously addition was represented by concatenating the line segments together, now the the line segments have directions, and therefore a beginning and an end. When concatenating these directed line segments, we start the second line segment at the end of the first one. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively cancelling part or all of it out.[7.NS.1b] Later in high school, if students encounter vectors, they will be able to see this as one-dimensional vector addition.

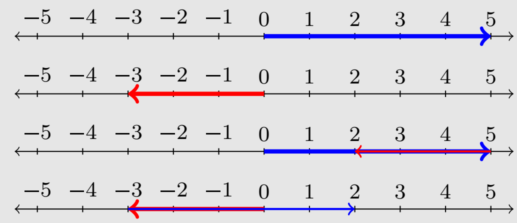
A fundamental fact about addition of rational numbers is that $p + (-p) = 0$ for any rational number p; in fact, this is a new property of operations that comes into play when negative numbers are introduced. This property can be introduced using situations in which the equation makes sense in a context.[7.NS.1a] For example, the operation of adding an integer could be modeled by an elevator moving up or down a certain number of floors. It can also be shown using the directed line segment model of addition on the number, as shown in the margin.[7.NS.1b]

It is common to use colored chips to represent integers, with one color representing positive integers and another representing negative integers, subject to the rule that chips of different colors cancel each other out; thus, a number is not changed if you take away or add such a pair. This is rather a different representation than the number line. On the number line, the equation $p+(-p) = 0$ follows from the definition of addition using directed line segments. With integer chips, the equation $p + (-p) = 0$ is true by definition since it it is encoded in the rules for manipulating the chips. Also implicit in the use of chips is that the commutative and associative properties extend to addition of integers, since combining chips can be done in any order.

However, the integer chips are not suited to representing rational numbers that are not integers. Whether such chips are used or not, the Standards require that students eventually understand location and addition of rational numbers on the number line. With the number line model, showing that the properties of operations extend to rational numbers requires some reasoning. Although it is not appropriate in Grade 6 to insist that all the properties be proved proved to hold in the number line or chips model, experimenting with them in these models is a good venue for reasoning (MP.2).[7.NS.1d]

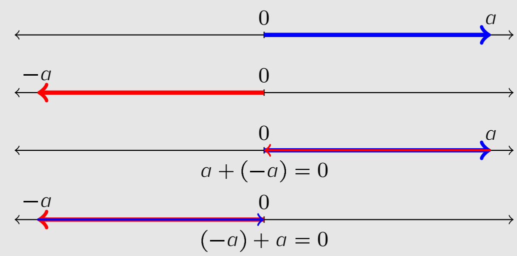
Subtraction of rational numbers is defined the same way as for positive rational numbers: $p-q$ is defined to be the missing addend in $q + ? = p$. For example, in earlier grades, students understand $5 - 3$ as the missing addend in $3 + ? = 5$. On the number line, it

Showing $5 + (-3) = 2$ and $-3 + 5 = 2$ on the number line



The number 5 is represented by the blue arrow pointing right from 0, and the number −3 is represented by the red arrow pointing left from 0. To add $5 + (-3)$ we place the arrow for 5 down first then attach the arrow for −3 to its endpoint. To add $-3 + 5$ we place the arrow for −3 down first then attach the arrow for 5 to its endpoint.

Showing $a + (-a) = 0$, and $(-a) + a = 0$ on the number line



[7.NS.1] Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

a Describe situations in which opposite quantities combine to make 0.

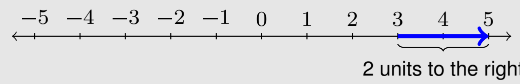
b Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

d Apply properties of operations as strategies to add and subtract rational numbers.

is represented as the distance from 3 to 5. Or, with our newfound emphasis on direction on the number line, we might say that it is how you get from 3 from 5; by going two units to the right (that is, by adding 2).

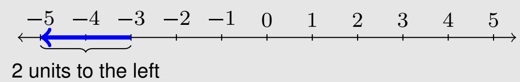
In Grade 6 students apply the same understanding to $(-5) - (-3)$. It is the missing addend in $(-3) + ? = -5$, or how you get from -3 to -5. Since -5 is two units to the left of -3 on the number line, the missing addend is -2.

Showing $5 - 3 = 2$ on the number line.



You get from 3 to 5 by adding 2, so $5 - 3 = 2$.

Showing $(-5) - (-3) = -2$ on the number line.



You get from -3 to -5 by adding -2, so $(-5) - (-3) = -2$

With the introduction of direction on the number line, there is a distinction between the distance from a and b and how you get from a to b. The distance from -3 to -5 is 2 units, but the instructions how to get from -3 to -5 are "go two units to the left." The distance is a positive number, 2, wherease "how to get there" is a negative number -2. In Grade 6 we introduce the idea of absolute value to talk about the size of a number, regardless of its sign. It is always a positive number or zero. If p is positive, then its absolute value $|p|$ is just p; if p is negative then $|p| = -p$. With this interpretation we can say that the absolute value of $p - q$ is just the distance from p to q, regardless of direction.[7.NS.1c]

Understanding $p - q$ as a missing addend also helps us see that $p + (-q) = p - q$.[7.NS.1c] Indeed, $p - q$ is the missing number in

$$q + ? = p$$

and $p + (-q)$ satisfies the description of being that missing, number:

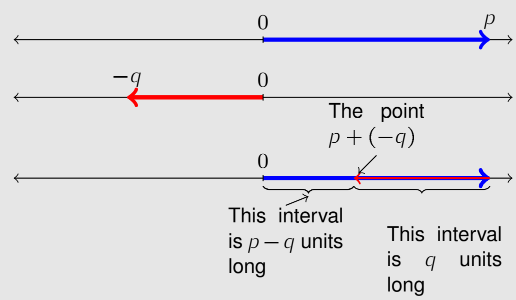
$$q + (p + (-q)) = p + (q + (-q)) = p + 0 = p.$$

The figure in the margin illustrates this in the case where p and q are positive and $p > q$.

7.NS.1c Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

c Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Showing $p + (-q) = p - q$ on the number line



The red directed interval representing $-q$ is q units long, so the remaining part of the blue directed interval representing p is $p - q$ units long.

Multiplication and division of rational numbers Hitherto we have been able to come up with visual models to represent rational numbers, and the operations of addition and subtraction on them. This starts to break down with multiplication and division, and students must rely increasingly on the properties of operations to build the necessary bridges from their previous understandings to situations where one or more of the numbers might be negative.

For example, multiplication of a negative number by a positive whole number can still be understood as before; just as 5×2 can be understood as $2 + 2 + 2 + 2 + 2 = 10$, so 5×-2 can be understood as $(-2) + (-2) + (-2) + (-2) + (-2) = -10$. We think of 5×2 as five jumps to the right on the number line, starting at 0, and we think of $5 \times (-2)$ as five jumps to the left.

But what about $\frac{3}{4} \times -2$, or -5×-2? Perhaps the former can be undersood as going $\frac{3}{4}$ of the way from 0 to -2, that is $-\frac{3}{2}$. For the latter, teachers sometimes come up with metaphors involving going backwards in time or repaying debts. But in the end these metaphors do not explain why $-5 \times -2 = 10$. In fact, this is a choice we make, not something we can justify by appeals to real world situations.

Why do we make the choice of saying that a negative times a negative is positive? Because we want to extend the operation of multiplication to rational number is such a way that *all* of the properties of operations are satisfied.[7.NS.2a] In particular, the property that really makes a difference here is the distributive property. If you want to be able to say that

$$4 \times (5 + (-2)) = 4 \times 5 + 4 \times (-2),$$

you have to say that $4 \times (-2) = -8$, because the number on the left is 12 and the first addend on the right is 20. This leads to the rules

$$\text{positive} \times \text{negative} = \text{negative} \quad \text{and} \quad \text{negative} \times \text{positive} = \text{negative}.$$

If you want to be able to say that

$$(-4) \times (5 + (-2)) = (-4) \times 5 + (-4) \times (-2),$$

then you have to say that $(-4) \times (-2) = 8$, since now we know that the number on the left is -12 and the first addend on the right is -20. This leads to the rule

$$\text{negative} \times \text{negative} = \text{positive}.$$

Why is it important to maintain the distributive property? Because when students get to algebra, they use it all the time. They must be able to say $-3x - 6y = -3(x + 2y)$ without worrying about whether x and y are positive or negative.

The rules about moving negative signs around in a product result from the rules about multiplying negative and positive numbers. Think about the various possibilities for p and q in

$$p \times (-q) = (-p) \times q = -pq.$$

If p and q are both positive, then this just a restatement of the rules above. But it still works if, for example, p is negative and q is positive. In that case it says

$$\text{negative} \times \text{negative} = \text{positive} \times \text{positive} = \text{positive}.$$

Just as the relationship between addition and subtraction helps students understand subtraction of rational numbers, so the relationship between multiplication and division helps them understand division. To calculate $-8 \div 4$, students recall that $(-2) \times 4 = -8$, and so $-8 \div 4 = -2$. By the same reasoning,

$$-8 \div 5 = -\frac{8}{5} \quad \text{because} \quad -\frac{8}{5} \times 5 = -8.$$

7.NS.2a Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

a Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

This means it makes sense to write

$$-8 \div 5 \quad \text{as} \quad \frac{-8}{5}.$$

Until this point students have not seen fractions where the numerator or denominator could be a negative integer. But working with the corresponding multiplication equations allows students to make sense of such fractions. In general, they see that[7.NS.2b]

$$-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$$

for any integers p and q, with $q \neq 0$.

Again, using multiplication as a guide, students can extend division to rational numbers that are not integers.[7.NS.2c] For example

$$\frac{2}{3} \div (-\frac{1}{2}) = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

And again it makes sense to write this division as a fraction:

$$\frac{\frac{2}{3}}{-\frac{1}{2}} = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

Note that this is an extension of the fraction notation to a situation it was not originally designed for. There is no sense in which we can think of

$$\frac{\frac{2}{3}}{-\frac{1}{2}}$$

as $\frac{2}{3}$ parts where one part is obtained by dividing the line segment from 0 to 1 into $-\frac{1}{2}$ equal parts! But the fact that the properties of operations extend to rational numbers means that calculations with fractions extend to these so-called complex fractions $\frac{p}{q}$, where p and q could be rational numbers, not only integers. By the end of Grade 7, students are solving problems involving complex fractions.[7.NS.3]

Decimals are special fractions, those with denominator 10, 100, 1000, etc. But they can also be seen as a special sort of measurement on the number line, namely one that you get by partitioning into 10 equal pieces. In Grade 7 students begin to see this as a possibly infinite process. The number line is marked off into tenths, each of which is marked off into 10 hundreths, each of which is marked off into 10 thousandths, and so on ad infinitum. These finer and finer partitions constitute a sort of address system for numbers on the number line: 0.635 is, first, in the neighborhood betweeen 0.6 and 0.7, then in part of that neighborhood between 0.63 and 0.64, then exactly at 0.635.

The finite decimals are the rational numbers that eventually come to fall exactly on one of the tick marks in this decimal address system. But there are numbers that never come to rest, no

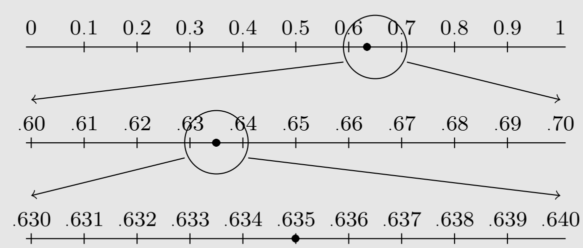
7.NS.2b Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

b Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.

c Apply properties of operations as strategies to multiply and divide rational numbers.

7.NS.3 Solve real-world and mathematical problems involving the four operations with rational numbers.

Zooming in on 0.635



The finite decimal 0.635 *can eventually be found sitting one of the tick marks at the thousandths level.*

matter how far down you go. For example, $\frac{1}{3}$ is always sitting one third of the way along the third subdivision. It is 0.33 plus one-third of a thousandth, and 0.333 plus one-third of a ten thousandth, and so on. The decimals 0.33, 0.333, 0.3333 are successively closer and closer approximations to $\frac{1}{3}$. We summarize this situation by saying that $\frac{1}{3}$ has an infinite decimal expansion consisting entirely of 3s

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3},$$

where the bar over the 3 indicates that it repeats indefinitely. Although a rigorous treatment of this mysterious infinite expansion is not available in middle school, students in Grade 7 start to develop an intuitive understanding of decimals as a (possibly) infinite address system through simple examples such as this.[7.NS.2d]

For those rational numbers that have finite decimal expansions, students can find those expansions using long division. Saying that a rational number has a finite decimal expansion is the same as saying that it can be expressed as a fraction whose numerator is a base-ten unit (10, 100, 1000, etc.). So if $\frac{a}{b}$ is a fraction with a finite expansion, then

$$\frac{a}{b} = \frac{n}{10} \quad \text{or} \quad \frac{n}{100} \quad \text{or} \quad \frac{n}{1000} \quad \text{or} \quad \ldots,$$

for some whole number n. If this is the case, then

$$\frac{10a}{b} = n \quad \text{or} \quad \frac{100a}{b} = n \quad \text{or} \quad \frac{1000a}{b} = n \quad \text{or} \quad \ldots.$$

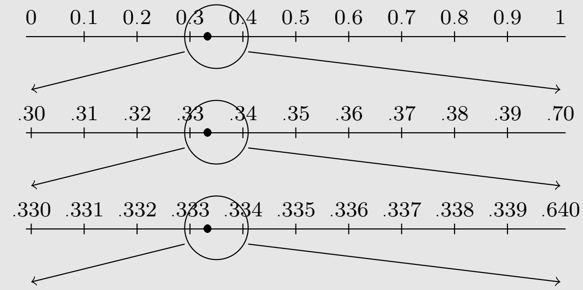
So we can find the whole number n by dividing b successively into $10a$, $100a$, $1000a$, and so on until there is no remainder.[7.NS.2d] The margin illustrates this process for $\frac{3}{8}$, where we find that there is no remainder for the division into 3000, so

$$3000 = 8 \times 375,$$

which means that

$$\frac{3}{8} = \frac{375}{1000} = 0.375.$$

Zooming in on $\frac{1}{3}$

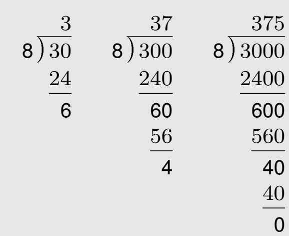


The fraction $\frac{1}{3}$ is never eventually on one of the tick marks. It is always one third the way along the third subdivision.

7.NS.2d Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

d Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Division of 8 into 3 times a base-ten unit



Notice that it is not really necessary to restart the division for each new base-ten unit, since the steps from the previous calculation carry over to the next one.

Grade 8

Know that there are numbers that are not rational, and approximate them by rational numbers In Grade 7 students encountered infinitely repeating decimals, such as $\frac{1}{3} = 0.\overline{3}$. In Grade 8 they understand why this phenomenon occurs, a good exercise in expressing regularity in repeated reasoning (MP8).[8.NS.1] Taking the case of $\frac{1}{3}$, for example, we can try to express it as a finite decimal using the same process we used for $\frac{3}{8}$ in Grade 7. We successively divide 3 into 10, 100, 1000, hoping to find a point at which the remainder is zero. But this never happens; there is always a remainder of 1. After a few tries it is clear that the long division will always go the same way, because the individual steps always work the same way: the next digit in the quotient is always 3 resulting in a reduction of the dividend from one base-unit to the next smaller one (see margin). Once we have seen this regularity, we see that $\frac{1}{3}$ can never be a whole number of decimal base-ten units, no matter how small they are.

A similar investigation with other fractions leads to the insight that there must always eventually be a repeating pattern, because there are only so many ways a step in the algorithm can go. For example, considering the possibility that $\frac{2}{7}$ might be a finite decimal with, we try dividing 7 into 20, 200, 2000, etc., hoping to find a point where the remainder is zero. But something happens when we get to dividing 7 into 2,000,000, the left-most division in the margin. We find ourselves with a remainder of 2. Since we started with a numerator of 2, the algorithm is going to start repeating the sequence of digits from this point on. So we are never going to get a remainder of 0. All is not in vain, however. Each calculation gives us a decimal approximation of $\frac{2}{7}$. For example, the left-most calculation in the margin tells us that

$$\frac{2}{7} = \frac{1}{1000000}\frac{2000000}{7} = 0.285714 + \frac{2}{7} \times 0.0000001,$$

and the next two show that

$$\begin{aligned} \frac{2}{7} &= 0.2857142 + \frac{6}{7} \times 0.00000001 \\ \frac{2}{7} &= 0.28571428 + \frac{4}{7} \times 0.000000001. \end{aligned}$$

Noticing the emergence of the repeating pattern 285714 in the digits, we say that

$$\frac{2}{7} = 0.\overline{285714}.$$

The possibility of infinite repeating decimals raises the possibility of infinite decimals that do not ever repeat. From the point of view of the decimal address system, there is no reason why such number should not correspond to a point on the number line. For

[8.NS.1] Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Division of 3 into 100, 1000, and $10{,}000$

33	333	3333
3) 100	3) 1000	3) 10000
90	900	9000
10	100	1000
9	90	900
1	10	100
	9	90
	1	10
		9
		1

Repeated division of 3 into larger and larger base ten units shows the same pattern.

Division of 7 into multiples of 2 times larger and larger base-ten units

285714	2857142	28571428
7) 2000000	7) 20000000	7) 200000000
1400000	14000000	140000000
600000	6000000	60000000
560000	5600000	56000000
40000	400000	4000000
35000	350000	3500000
5000	50000	500000
4900	49000	490000
100	1000	10000
70	700	7000
30	300	3000
28	280	2800
2	20	200
	14	140
	6	60
		56
		4

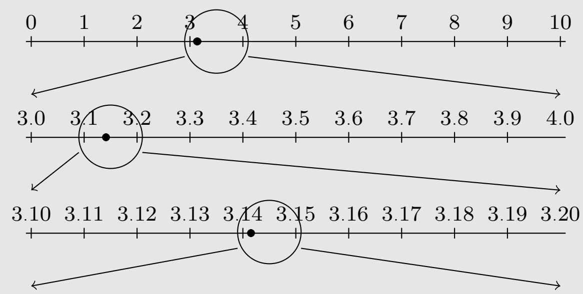
The remainder at each step is always a single digit multiple of a base-ten unit so eventually the algorithm has to cycle back to the same situation as some earlier step. From then on the algorithm produces the same sequence of digits as from the earlier step, ad infinitum.

example, the number π lives between 3 and 4, and between 3.1 and 3.2, and between 3.14 and 3.15, and so on, with each successive decimal digit narrowing its possible location by a factor of 10.

Numbers like π, which do not have a repeating decimal expansion and therefore are not rational numbers, are called *irrational*.[8.NS.1] Although we can calculate the decimal expansion of π to any desired accuracy, we cannot describe the entire expansion because it is infinitely long, and because there is no pattern (as far as we know). However, because of the way in which the decimal address system narrows down the interval in which a number lives, we can use the first few digits of the decimal expansion to come up with good decimal approximations of π, or any other irrational number. For example, the fact that π is between 3 and 4 tells us that π^2 is between 9 and 16; the fact that π is between 3.1 and 3.2 tells us that π^2 is between 9.6 and 10.3, and so on.[8.NS.2]

8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Zooming in on π



The number π has an infinite non-repeating decimal expansion which determines each successive sub-interval to zoom in on.

8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2).

High School, Number*

The Real Number System

Extend the properties of exponents to rational exponents In Grades 6–8 students began to widen the possible types of number they can conceptualize on the number line. In Grade 8 they glimpse the existence of irrational numbers such as $\sqrt{2}$. In high school, they start a systematic study of functions that can take on irrational values, such as exponential, logarithmic, and power functions. The first step in this direction is the understanding of numerical expressions in which the exponent is not a whole number. Functions such as $f(x) = x^2$, or more generally polynomial functions, have the property that if the input x is a rational number, then so is the output. This is because their output values are computed by basic arithmetic operations on their input values. But a function such as $f(x) = \sqrt{x}$ does not necessarily have rational output values for every rational input value. For example, $f(2) = \sqrt{2}$ is irrational even though 2 is rational.

The study of such functions brings with it a need for an extended understanding of the meaning of an exponent. Exponent notation is a remarkable success story in the expansion of mathematical ideas. It is not obvious at first that a number such as $\sqrt{2}$ can be represented as a power of 2. But reflecting that

$$\left(\sqrt{2}\right)^2 = 2$$

and thinking about the properties of exponents, it is natural to define

$$2^{\frac{1}{2}} = \sqrt{2}$$

since if we follow the rule $(a^b)^c = a^{bc}$ then

$$\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2.$$

Similar reasoning leads to a general definition of the meaning of a^b whenever a and b are rational numbers.[N-RN.1] It should be noted high school mathematics does not develop the mathematical ideas necessary to prove that numbers such as $\sqrt{2}$ and $3^{\frac{1}{5}}$ actually exist;

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

*This progression concerns Number and Quantity standards related to number. The remaining standards are discussed in the Quantity Progression.

in fact all of high school mathematics depends on the fundamental assumption that properties of rational numbers extend to irrational numbers. This is not unreasonable, since the number line is populated densely with rational numbers, and a conception of number as a point on the number line gives reassurance from intuitions about measurement that irrational numbers are not going to behave in a strangely different way from rational numbers.

Because rational exponents have been introduced in such a way as to preserve the laws of exponents, students can now use those laws in a wider variety of situations. For example, they can rewrite the formula for the volume of a sphere of radius r,

$$V = \frac{4}{3}\pi r^3,$$

to express the radius in terms of the volume,[N-RN.2]

$$r = \left(\frac{3}{4}\frac{V}{\pi}\right)^{\frac{1}{3}}.$$

N-RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers An important difference between rational and irrational numbers is that rational numbers form a number system. If you add, subtract, multiply, or divide two rational numbers, you get another rational number (provided the divisor is not 0 in the last case). The same is not true of irrational numbers. For example, if you multiply the irrational number $\sqrt{2}$ by itself, you get the rational number 2. Irrational numbers are defined by not being rational, and this definition can be exploited to generate many examples of irrational numbers from just a few.[N-RN.3] For example, because $\sqrt{2}$ is irrational it follows that $3+\sqrt{2}$ and $5\sqrt{2}$ are also irrational. Indeed, if $3+\sqrt{2}$ were an irrational number, call it x, say, then from $3+\sqrt{2} = x$ we would deduce $\sqrt{2} = x-3$. This would imply $\sqrt{2}$ is rational, since it is obtained by subtracting the rational number 3 from the rational number x. But it is not rational, so neither is $3+\sqrt{2}$.

N-RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Although in applications of mathematics the distinction between rational and irrational numbers is irrelevant, since we always deal with finite decimal approximations (and therefore with rational numbers), thinking about the properties of rational and irrational numbers is good practice for mathematical reasoning habits such as constructing viable arguments and attending to precisions (MP.3, MP.6).

Complex Numbers

That complex numbers have a practical application is surprising to many. But it turns out that many phenomena involving real numbers become simpler when the real numbers are viewed as a subsytem of the complex numbers. For example, complex solutions of differential equations can give a unified picture of the behavior of real solutions. Students get a glimpse of this when they study complex solutions of quadratic equations. When complex numbers are brought into the picture, every quadratic polynomial can be expressed as a product of linear factors:

$$ax^2 + bx + c = a(x - r)(x - s).$$

The roots r and s are given by the quadratic formula:

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

When students first apply the quadratic formula to quadratic equations with real coefficients, the square root is a problem if the quantity $b^2 - 4ac$ is negative. Complex numbers solve that problem by introducing a new number, i, with the property that $i^2 = -1$, which enables students to express the solutions of any quadratic equation.[N-CN.7]

N-CN.7 Solve quadratic equations with real coefficients that have complex solutions.

One remarkable fact about introducing the number i is that it works: the set of numbers of the form $a + bi$, where $i^2 = -1$ and a and b are real numbers, forms a number system. That is, you can add, subtract, multiply and divide two numbers of this form and get another number of the same form as the result. We call this the system of complex numbers.[N-CN.1]

N-CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

All you need to perform operations on complex numbers is the fact that $i^2 = -1$ and the properties of operations.[N-CN.2] For example, to add $3 + 2i$ and $-1 + 4i$ we write

N-CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

$$(3 + 2i) + (-1 + 4i) = (3 + -1) + (2i + 4i) = 2 + 6i,$$

using the associative and commutative properties of addition, and the distributive property to pull the i out, resulting in another complex number. Multiplication requires using the fact that $i^2 = -1$:

$$(3 + 2i)(-1 + 4i) = -3 + 10i + 8i^2 = -3 + 10i - 8 = -11 + 10i.$$

Division of complex numbers is a little tricker, but with the discovery of the complex conjugate $a - bi$ we find that every non-zero complex number has a multiplicative inverse.[N-CN.3] If at least one of a and b is not zero, then

N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

$$(a + bi)^{-1} = \frac{1}{a^2 + b^2}(a - bi)$$

because

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Students who continue to study geometric representations of complex numbers in the complex plane use both rectangular and polar coordinates which leads to a useful geometric interpretation of the operations.[N-CN.4, N-CN.5] The restriction of these geometric interpretations to the real numbers yields and interpretation of these operations on the number line.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

One of the great theorems of modern mathematics is the Fundamental Theorem of Algebra, which says that every polynomial equation has a solution in the complex numbers. To put this into perspective, recall that we formed the complex numbers by creating a solution, i, to just one special polynomial equation, $x^2 = -1$. With the addition of this one solution, it turns out that every polynomial equation, for example $x^4 + x^2 = -1$, also acquires a solution. Students have already seen this phenomenon for quadratic equations.[N-CN.9]

N-CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Although much of the study of complex numbers goes beyond the college and career ready threshold, as indicated by the (+) on many of the standards, it is a rewarding area of exploration for advanced students.